THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
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THE MATHEMATICAL PROBLEMS OF AERODYNAMICS.*

By Professor H. Levy, M.A., D.Sc.

The equations governing the motion of a viscous fluid were first obtained by Navier more than 100 years ago ("Memoire sur les Lois du Mouvement des Fluides," *Mem. de l'Acad. des Sciences*, vi. 389, 1822), and in spite of their close study by Stokes, Helmholtz, Kelvin, Rayleigh, Lamb and numerous other mathematicians of great eminence, no complete unrestricted solution for any case of practical importance has yet been discovered. As they stand, the mathematical difficulties presented by the equations have so far been found to be too formidable, and whatever progress has been achieved has been by imposing restrictions on the form of the equations, and therefore serious limitations on the nature of the fluid motion studied. It may be that the mathematical symbolism in the formulation of the problem of viscous fluid flow as usually presented is not that best adapted for its purpose, that it is not as natural a medium of expression as for example Tensor Analysis is for Relativity; that in fact the essential factors that govern the eddying, for instance, in the wake of a moving body are not presented as governing the structure of the equations. Whatever new method of approach may appear in the future, at the moment attempts to apply Hydrodynamic and Aerodynamic considerations to problems of aerial flight must on the mathematical side be directed through the fundamental equations as here set out:

$$\left.\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v,\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 w,\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0.
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

That approach is likely to be considerably facilitated if, as is the case in most other branches of Mathematical Physics, physical or geometrical considerations may lead us to anticipate solutions of a special character, periodic in space and decaying time for example, or to expect that the value of certain constants or terms in the equation are crucial in determining the nature of the solution. This is the case with the hydrodynamical equations where a

* A Lecture delivered at the Annual Meeting of the Mathematical Association, Jan. 7th, 1929.

certain constant, known as Reynolds' number, appears to play a rôle analogous to that taken by the characteristic number or *Eigenwert* in other branches of mathematical physics. Initially this may be seen at once from the fundamental equations. If the problem be that of the steady motion V of a body of standard length l of given shape, then all the terms that occur in the equation may be thrown into nondimensional form by the simple transformation:

$$(x, y, z)=l(x', y', z'), \quad (uvw)=V(u', v', w'),$$
$$t=\frac{l}{V}t', \quad \frac{p}{\rho V^2 l}=p'.$$

The equations of motion then become relations not between physical quantities but between mere variable numbers. Dropping the accents attached to the letters the equations take the form

$$\left.\begin{matrix} R.\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial p}{\partial x}\right)=\nabla^2 u, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{matrix}\right\} \quad \text{.....................(2)}$$

where $R=Vl/\nu=$Reynolds' number, and it is obvious that the motion is likely to be governed by the magnitude of the terms on one or other side of the equations according as R is very large or very small. The orthodox steady solutions of these equations to be found in the text-books, for example steady flow between parallel walls or along a straight pipe, correspond to values of $u\ v\ w$ and p that reduce the two sides of these equations separately to zero and therefore the characteristics of the motion that may be expected to, and do, arise at varying values of R are not present in these solutions. The problem in this respect is closely analogous to that which occurs in elastic structures under load. If a thin strut be subjected to a purely longitudinal end thrust F, the undeflected position will correspond to a solution of the equations of position for all values of this end thrust, nevertheless there does exist critical values of F corresponding to characteristic numbers or "eigenwerte" of the equations for which solutions other than the undeflected one, may arise—the so-called Euler loads. The corresponding mathematical problem in Aerodynamics, and almost the key problem in the theoretical development of that subject, centres itself round the determination of the critical value or values of $R=Vl/\nu$ for which solutions other than the uniform steady motions exist. The problem is accordingly seen to be one of a purely mathematical nature, without necessary reference to its aerodynamic interpretation, and one to which pure mathematicians might profitably devote attention. In one special case only has this problem been solved, but to this we shall return later.

The crucial part played by Vl/ν may be recognised from very simple considerations without reference to the equations of motion. This may be done in two ways. In the first place, regard the fluid as of molecular constitution with mean molecular velocity$=c$, $m=$mass/sec crossing unit area$=\rho c$, $L=$mean free path. But from simple considerations of momentum transfer it is well known that the viscosity

$$\mu \propto mL \propto \rho c L, \quad \nu=\frac{\mu}{\rho}=\text{kinematic viscosity};$$

$$\therefore\ \frac{Vl}{\nu}=\frac{Vl\rho}{\mu}\propto\frac{Vl\rho}{\rho c L}\propto\frac{V}{c}\times\frac{l}{L}=\frac{\text{veloc. of body}}{\text{veloc. of mol.}}\times\frac{\text{length of body}}{\text{mean free path}}.$$

(i) *Range for Brownian Movements.*

If $\frac{V}{c}\ll 1$, while the dimensions of the body are of the same order as the mean free path, then such small values of $\frac{Vl}{\nu}$ represent a range applicable,

say, to Brownian movements. Even Stokes' resistance formula for a sphere is not applicable in this range of Vl/ν.

(ii) *Aerodynamic Range.*

$$\frac{V}{c} \ll 1, \quad \frac{l}{L} \gg 1.$$

For this range we note that

$$\frac{Vl}{\nu} \propto \frac{1}{2}\rho V^2 \div \mu\left(\frac{V}{l}\right) \propto \frac{\text{dynamic pressure}}{\text{viscous drag}}.$$

For small values of $\frac{Vl}{\nu}$ in this range the state of motion is therefore determined by viscous action—as in the case of Stokes' linear resistance law for a sphere—but as $\frac{Vl}{\nu}$ increases dynamical forces preponderate, unsteady vortex motion and an approximately quadratic law of resistance rapidly develop—and turbulent motion sets in.

What are the values of $\frac{Vl}{\nu}$ at which these changes in type of motion set in? The orthodox solutions of the equation naturally give no clue to this question.

(iii) *Ballastic Region.*

$$\frac{V}{c} \approx 1, \quad \frac{l}{L} \gg 1.$$

The velocity of the body is of the order of that of sound, and the elastic properties of the gas come into play.

A second line of approach involving highly important practical implications is provided through a simple application of dimensional theory. It lays down in fact the basis on which the greater part of aerodynamic experimental research rests, and indicates how the results so derived are to be applied to the full scale problem presented by the aeroplane and airship. If, as before, V and l be the speed and standard dimension of a body of given shape moving through a viscous fluid of density ρ and kinematic viscosity ν, then by mere consideration of dimensions it can be shown that the resistance F of the body must in general be of the form

$$F = \rho V^2 l^2 f(Vl/\nu),$$

the speed at any point of the fluid fixed relatively to the body will be

$$v = V \,.\, g(Vl/\nu),$$

and the direction of motion at that point

$$\theta = h(Vl/\nu),$$

where the functions f, g and h can depend only on the shape of the body.

It follows that if two bodies of identical shape but different sizes, an aeroplane wing and its model for example, be moving through two fluids differing in density and viscosity, then if their speeds are so adjusted that the number $\frac{Vl}{\nu}$ is the same for both,

(i) The nature of the stream lines in the neighbourhood of the one body will be geometrically identical with that of the other. If at some value of Vl/ν the motion changes from steadiness to periodic motion or to turbulence in the one case, exactly the same thing will occur in the other at that value of Vl/ν.

(ii) $F/\rho V^2 l^2$ will be the same for both bodies. Thus if a chart for the model can be derived experimentally, showing the variation of $F/\rho V^2 l^2$ with Vl/ν, that chart will be equally applicable to the full scale. It is this consideration

that enables the results of model aerodynamic experiments to be applied to the determination of the forces, and therefore of the performance, of the full scale machine. In the past the practical difficulty has been encountered that with a model in a wind channel a sufficiently wide range of Vl/ν could not be obtained to reach to that for the full scale, but this has recently been overcome by a special form of wind channel in which the air is under high pressure. The consequent effect on ν is such as to enable the full scale Vl/ν to be adequately attained.

The validity of this simple analysis has been thoroughly tested by Stanton in certain, now classical experiments, where the surface friction on pipes of various diameters through which fluids were passed (air and water) was measured. This was carried through over a wide range of Vl/ν. The chart produced by plotting the equivalent of $F/\rho V^2l^2$ against Vl/ν showed all the points irrespective of the nature of the fluid, its speed, or the pipe diameter—to lie on the one characteristic curve and indicated clearly by the rapid change in the nature of this curve at a certain position the specific value of Vl/ν at which the flow passed from the streaming to the turbulent stage. This work of Stanton may be regarded as the natural development of the pioneer work of Osborne Reynolds, who in a series of beautiful experiments first demonstrated the existence of a critical value of Vl/ν.

The natural impulse of the mathematician when faced with a set of non-linear differential equations to which he is called upon to find a solution is, if possible, to devise a justification for ignoring the awkward non-linear terms in his equation; for the study of non-linear equations is an almost virgin field.

Such a justification in respect to the hydrodynamical equations is to be found in certain branches of wave motion, but these simplifications are not so readily invoked in aerodynamic application without severe restrictions on the validity of the conclusions.

If we examine the differential equations of motion in the non-dimensional form, it is obvious that the neglect of the product terms $u\dfrac{\partial u}{\partial x}$, etc., will be legitimate when Reynolds' number $\dfrac{Vl}{\nu}$ is itself exceedingly small. In this case we are restricted to the study of the forces on bodies at values of Vl/ν so far below the critical that little of interest can be gleaned as regards the nature and origin of eddying and turbulence which is such a prominent feature of the motion associated with aerodynamic phenomena proper. Such a case is the classical solution by Stokes of the resistance of a sphere to slow motion, fruitful enough in its application to other domains of physics, but of little practical interest in theoretical aerodynamics. A natural extension suggested by the writer in this connection is, instead of neglecting the product terms, to seek for solutions which are power series in $R = Vl/\nu$, the coefficients of which are functions of position. The first term in such an expansion is clearly the solution just mentioned, corresponding in effect to Vl/ν zero. The determination of the coefficients of the higher powers of Vl/ν becomes exceedingly cumbersome and formidable. In any case it is not clear that an expansion of this nature would be necessarily convergent over a reasonably large range of values of Vl/ν. The subject has not, however, been adequately investigated. Here, indeed, is another problem for the pure mathematician. In what circumstances will a solution of the fundamental equations (2) exist as a series in ascending powers of Vl/ν, and when such a solution exists, for what range of values of Vl/ν will it be convergent? If one may hazard a guess on a matter not particularly amenable to guesswork, it would be that the upper bound of this range of Vl/ν where it exists marks the transition point from the state of smooth flow associated with slow viscous drag to sinuous motion and possibly the beginnings of eddying. A few attempts have been made to

determine the state of flow in the neighbourhood of bodies of simple geometrical shape—circular cylinders and flat plates—by numerical and graphical methods, beginning as a first approximation with the case of $Vl/\nu=0$, but so far the solutions have not been pressed to sufficiently high values of $\frac{Vl}{\nu}$ to indicate the commencement of eddying motion.

In the main this approach to the problem has really been concerned with the attempt to derive expressions for the law of resistance to motion, and not directly with the determination of the critical value of Vl/ν at which the flow and the law of resistance changes rapidly in type. Ample experimental evidence exists to prove that the critical is in most cases not sharp and definite. By the avoidance of disturbance in the motion of fluid in a pipe, for example, it is possible to reach a value of Vl/ν considerably above that at which turbulence begins for cases where these precautions are not adopted. It would appear from these considerations that when turbulence originates in an extended fluid it does so from the fact that the steady motion at this value of Vl/ν has become unstable, if not to small disturbances at least to large. Thus at this stage two solutions of the equations widely differing in nature would appear to be possible. A number of writers (Rayleigh, Lorentz, Orr) have approached this question from the point of view of the disturbance by examining in what circumstances the total energy of an arbitrary disturbance will increase or diminish. The results although of interest are not conclusive. In the case of disturbed flow (u, v) in a uniform channel for instance, moving in the undisturbed state with a velocity $U(y)$, they lead to the conclusion that the rate of gain of total energy of the disturbance will be *negative* if $\frac{Vl}{\nu}$ is less than the maximum value of R which makes

$$\iint\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)^2 dx\,dy+R\iint\frac{dU}{dy}\,uv\,dx\,dy$$

zero for arbitrary choice of u and v; that is to say, the motion could not be unstable for a lower value of Vl/ν than this.

Thus the problem resolves itself into one of a purely mathematical nature in the calculus of variations—to determine the maximum value of

$$\frac{\iint\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)^2 dx\,dy}{\iint\frac{dU}{dy}\,.\,uv\,dx\,dy}$$

where $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ and u and v are zero over the boundaries.

If the problem is that of one plane shearing over another,

$$U=y \text{ and } (u, v)=0 \text{ when } y=0 \text{ and } y=1,$$

the region of integration extending over the field between $y=0$ and $y=1$.

Lorentz and Orr, by different methods implying however special types of disturbance, have applied this criterion to the case of the shearing motion of one plane over another, and have derived values of 44 and 177 respectively for Vl/ν, but no complete analysis giving the most "dangerous" disturbance, as it were, has yet been forthcoming. Even if it were, what would at best be provided would be a value of Reynolds' number such that for all smaller values the motion would certainly be stable to infinitesimal disturbances. It would not necessarily provide the "critical" value, which is known to be somewhere in the region of $R=1000$, and possibly corresponds to a state of motion widely different from the steady state.

With one exception, a direct attack on the equations of motion slightly disturbed from the steady state has yielded nothing but disappointment.

Kelvin in 1884 for the Adams' Prize produced what purported to be a complete analysis of the stability of viscous flow between parallel walls demonstrating its stability for all values of Vl/ν. His "proof" has not survived the fundamental criticisms of Rayleigh and Orr. The exception referred to is the solution provided by Professor G. I. Taylor of the state of disturbed motion between two concentric cylinders spinning at different speeds. He has shown mathematically—involving rather formidable calculations—that in certain circumstances there will develop a series of vortex rings lying between the cylinders like anchor rings whose axis coincides with the axis of the cylinder. The fluid composing neighbouring rings rotates in opposite directions; and the whole analysis, little as the direct bearing of this particular problem has on aerodynamic questions, provides interesting evidence on the manner of formation of vortices in a disturbed fluid. An experimental investigation of this same problem has amply verified Taylor's results, but it shows in addition that in certain circumstances, instead of vortex rings being formed, a spiral vortex winds round the inner cylinder. Mathematically this solution has not yet been found.

A remarkable feature of what might be called the "newer hydrodynamics," developed under the impetus of aerodynamic research of an experimental nature, is the return to life of the widely discredited "perfect fluid," rejuvenated, seeking its place in the air. From the standpoint of the equations of motion this case arises by setting $\nu=0$, that is to say, the fluid is presumed devoid of viscosity, and the motion is that of an "electric fluid."

If real fluids ever approximate in their motion to that of a perfect fluid, it would be expected then to occur only at large values of Vl/ν or in those parts of the fluid and for such motions as are not likely to be affected by viscous forces. The immediate neighbourhood of the body is thus excluded from this region, since no matter how small the viscosity, there is no relative motion between the surface and the fluid in contact with it. In a perfect fluid, on the other hand, the liquid must be presumed to flow unhindered over the surface; there is "perfect" slip. At first sight this would appear to destroy any possibility of the application of the perfect fluid to the class of problem we are envisaging. After all it is just this interaction between body and fluid that one would expect would play an important part in the determination of resistance. If, moreover, surface conditions bear any relation to the problem of eddying (and it is usually considered that vorticity proceeds from the boundaries outwards through the fluid), it is not to be anticipated that the perfect fluid could fulfil any useful purpose in this connection.

It is easily proved, of course, that a body moving in a perfect fluid would experience no resistance, and although Kelvin has shown how in certain special circumstances a hollow vortex may be shed from the surface of a body in such a fluid, there is nothing to indicate how in the normal conditions in which eddying does arise this could occur. It can be proved in fact that the total vorticity in such a fluid, once established, remains constant. Irrotational motion always remains irrotational. And yet the peculiar fact remains that the pressure distribution around a surface as determined experimentally coincides almost exactly over a large part of the body, usually the forward position, with that predicted by calculation for that body in a perfect fluid. This has been verified over such a wide range of surfaces, cylinders of various shapes, ellipsoids, aerofoils, that it would appear to be more than a mere coincidence.

In the case of an airship model, for example, of good profile, *i.e.* of very low resistance, the distributions of normal pressure are almost identical except over a narrow band at the tail. The resistance therefore experienced by such a "stream-line" surface must be almost entirely due to "skin friction"—the integral of the tangential drag of the air on the surface. The explanation is to be sought in what has become known as the "Boundary Layer Theory"

due in the main to Professor Prandtl of Göttingen. According to him there is, enveloping any body, a blanket or skin of the fluid which is of such a nature that the surface of this layer in contact with the body has no motion relative to it, while the outer boundary of the layer is in motion with what would be practically the motion of a perfect fluid past a body of this shape. Thus the layer is one of rapid velocity gradient normally. As a statement of empiric fact for the forward position of the body at least there can be no question that this is very near to the truth. How it arises is of course a matter whose explanation is to be sought in the fundamental equations and their boundary conditions. On the assumption that such a layer exists, within which the velocity normal to the surface is small in comparison with the tangential velocity, and assuming also that the thickness of this layer is of the order of $\sqrt{\nu}$ where ν is the kinematic viscosity—an assumption that appears to be almost demanded by the equations, if the layer exists at all—Prandtl has established a modified set of equations of motions for two dimensions applicable to the boundary layer. They are

$$\left.\begin{aligned} q\frac{\partial q}{\partial s}+w\frac{\partial q}{\partial n}&=-\frac{1}{\rho}\frac{\partial p}{\partial s}+\nu\frac{\partial^2 q}{\partial n^2},\\ 0&=-\frac{1}{\rho}\frac{\partial p}{\partial n},\\ \frac{\partial q}{\partial s}+\frac{\partial w}{\partial n}&=0,\end{aligned}\right\}\quad\text{..........................(3)}$$

where q and w are the velocities in the directions of s and n respectively, n and s representing a system of normal and tangential curves. Since $\frac{\partial p}{\partial n}=0$, it is clear that wherever on the surface it is valid to assume the existence of a thin layer under approximately laminer flow, the normal pressure on the surface will be that of the fluid just outside the layer. This seems to argue, remembering the experimental facts, that such a layer exists at any rate round the forefront of the body.

By this semi-empiric stroke Prandtl has resolved the problem of viscous flow as governed by the fundamental equations (1) into two subsidiary problems, the treatment of the flow past the body on the assumption that the fluid is devoid of viscosity—a matter that can be handled with some measure of success—coupled with an analysis of the state of flow and the forces called into being in a thin layer adjacent to the body and governed by equations (3).

It is desirable to examine these two questions separately so that we may appreciate precisely the nature of the synthesis that has become effective in aerodynamical study. Consider in the first place the special manner in which the perfect fluid has been adapted for this purpose, especially in two-dimensional motion. For, it should be remarked, that for many of the important problems that arise two-dimensional motion is a close approximation to the actual flow. The motion past the central portion of a strut or a wing, for example, may be considered as affected only to a minor extent by the presence of the tips; in any case there are numerous empirical corrections of a simple nature that may be applied to allow for these deviations. For the steady two-dimensional motion of a non-viscous fluid with velocity V past a cylinder, the fundamental equations (2) take the simple form:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0,\quad u=-\frac{\partial\psi}{\partial x},\quad v=-\frac{\partial\psi}{\partial y},$$
$$p-p_0=\frac{1}{2}\rho(V^2-v^2),$$

The differential equation is, of course, that for conduction of heat and of electricity, and has been the subject of thorough study.

If $w=\phi+i\psi$ and $z=x+iy$, our problem will be solved when we have found the relation $w=f(z)$ which transforms the region outside the aerofoil or of the strut in the z plane into the upper half of the w plane subject to certain restrictions regarding the positions of any poles or singularities, and subject also to $\frac{dw}{dz}=V$ when $|z|\to\infty$, in order that the speed may be V at an infinite distance. Theoretically a solution can always be found; practically it is well nigh impossible to find a solution for a given shape of boundary. Fortunately for the special shapes required in aerodynamics an adequate transformation can be found. This, however, is all very formal hydrodynamics of the orthodox text-book sort. The special point that arises here, however, emerges from the fact that if a "circulation" be imposed on the motion round the aerofoil, instead of the resultant force being zero, there will be brought into being a force proportional to the strength of the circulation and to the velocity V of the fluid in a direction at right angles to V. This provides the appropriate lift on the aerofoil. The steps are as follows:

$$w=V\left(\zeta+\frac{1}{\zeta}\right)$$

represents the flow past a circular cylinder in the ζ-plane, the velocity at infinity being V.

$$w=V\left(\zeta+\frac{1}{\zeta}\right)+\frac{ik}{2\pi}\log\zeta$$

represents the flows past a circular cylinder on which a circulation of strength K has been imposed. For the moment K may be regarded as undetermined. The resultant lift at right angles to V will be

$$L=\rho VK.$$

If V be complex the flow at infinity will be inclined to the real axis. If instead of V we have $Ve^{i\alpha}$ for example, the flow at infinity will be inclined at an angle α.

It remains to transform the circular cylinder into the aerofoil shape. This it can easily be proved can be done by a transformation of the type

$$\zeta=z+\frac{a_1}{z}+\frac{a_2}{z^2}+\dots,$$

the coefficients being adjusted to make the singularities fall inside the boundary so that they may not correspond to infinite velocities in the fluid. The details of this transformation need not be entered into here; suffice it to say that it is easy to select transformations of this nature that correspond to aerofoil shapes and are very suitable for practical purposes. So far K the strength of the circulation is undetermined, but this is fixed from a very simple consideration. In general the stream line in the z plane which meets the body near the nose and subdivides to form the outline of the body coalesces again at some other point to form a stream line that passes off again to infinity. If the rear portion of the body is sharp as is approximately the case with aerofoils, then unless the point of bifurcation of the stream lines in the rear coincides with the sharp tail a position of infinite velocity will arise where the stream lines curve round the sharp trailing edge. The value of K, the strength of the circulation, is so adjusted as to make the point of bifurcation of the stream lines in the rear coincide with the sharp edge, thus eliminating the infinite velocity at that point. This is really a very simple matter to effect during the transformation. Moreover be it noted, by making V complex it was possible to alter the angle of approach of the fluid to the circular cylinder. Hence in the transformed motion the formula covers the case of varying angle of attack of fluid to the *same* aerofoil, and enables the force on the body at right angles to the "chord" to be calculated at various angles of attack.

The mathematical development is in the main due to Kutta and Joukowsky. The whole approach to this question is exceedingly pretty and ingenious, and really constitutes one of the most beautiful pieces of simple yet successful analysis to be found in this subject. To this has to be added the remarkable fact that the values of K, the circulation, as actually determined by experimental measurement of the air-velocity on certain contours surrounding model aerofoils in a wind tunnel, are almost exactly those required by the above theory, a dramatic and effective resurrection of a discredited fluid, into real life.

There are nevertheless certain severe limitations in its application. The contours from which K is checked experimentally have to be so selected that they cut the air in the wake of the aerofoil at right angles to the general direction of flow. Beyond an angle of attack quite definite and critical for each aerofoil, the wake is a region of disturbed flow, of turbulence and eddying, that finds no counterpart in the smooth streaming of the perfect flow. Thus the so-called angle of stall, the "burble" point, to use a technical term that has been borrowed from the Jabberwock, the critical position at which eddies begin to be thrown off in definite frequency and formation, in this theory at least finds no place.

Let us, however, at this stage return to the second of the two subsidiary problems—viz. that relating to the Prandtl boundary layer theory. Since the tangential velocity q ranges in a very narrow band from zero at the surface to the full value at the outer limit of the layer, $\frac{\partial q}{\partial n}$ must have a considerable value, and consequently this region must be one of high vorticity. Blasius has obtained a solution of the equations (3) applicable to flow over the surface of a flat plate lying in the direction of the flow. The frictional force he finds is proportional to $V^{\frac{3}{2}}$, which is roughly in accordance with experiment. The thickness of the layer according to his solution increases as $s^{\frac{1}{2}}$, measuring s from the nose of the plate, so that after some distance what might be called the local value of Reynolds' number for the layer may attain a value sufficiently high for turbulent motion to be set up even in the boundary layer. In actual fact the Blasius solution does begin to diverge seriously from the experimentally measured values a short distance backwards from the nose where presumably the vorticity flowing rearwards begins to accumulate. And here is seen a remarkable verification of the Prandtl theory. According to him the vorticity increasing in the layer as the fluid flows to the rear is finally discharged as discrete eddies forming a highly turbulent wake. Is it in actual fact the case that the vorticity constituting the eddy comes from this boundary layer or does it arise, say, from instability of flow in the rear of the body? In a series of very beautiful experiments Prandtl has proved his point conclusively.

Taking a number of cases of turbulent flow, through a divergent orifice, and past cylinders and spheres, he has shown that by drawing off the fluid that flows close to the boundary—tapping off the surface layer, in fact, *into* the body so as to reduce the disturbance of the fluid to a minimum—the violent eddying is completely eliminated, and the fluid flows smoothly and steadily around the surface exactly as would be expected from an inviscid fluid.

The problem of resistance, reduced as it was to a simple synthesis of that of the perfect fluid with circulation—providing the appropriate force normal to the general direction of flow—coupled with the interaction of the body with the boundary layer—providing the skin friction—is thus resolved into a study of the behaviour of the turbulent part of the layer—its extent along the surface and its intensity.

Bodies of very fine profile—speaking aerodynamically—such as airships, and nicely tapered struts, whose resistance is almost entirely skin-frictional,

are likely to reveal themselves as very sensitive to any disturbance that might extend the region of turbulence in the layer towards the direction of the nose. In point of fact it is known that this class of body, tested in wind tunnels where the incident air is artificially disturbed to varying extent, shows a varying resistance, possibly due to this cause. This is of considerable importance for two reasons: since a great part of aeronautical design is based directly on the result of air-tunnel test, it would imply in the first place that the condition of the air tunnel in this respect may be very vital; and we have to note in the second place a very considerable proportion of the exposed parts of aeroplanes and airships are of the shape referred to above.

Von Kármán has suggested another method of approach. It consists, not in a study of the origin of the eddying as it is produced by the interplay of viscosity and surface, but of a study of the wake after the eddying has set in. If, it is argued, the organised state of vortex motion in the wake can be examined mathematically, it should be possible from considerations of the momentum of the fluid far in front of the body and far behind it in the wake to determine an expression for the resistance experienced by the body, *i.e.* for the force exerted by the body in setting up the eddying motion. A photographic examination of the surface of a liquid in the wake of a moving cylinder reveals the interesting fact that the eddies are discharged in alternation from each edge, and rotating as if drawn in towards the rear of the body and appearing as dimples on the surface, they adopt a definite ordered arrangement. This consists of two parallel trails of oppositely rotating vortices, the members of the one row being situated opposite points midway between the members of the other row, the whole constituting what has become known as a "vortex street."

An investigation of the stability of such an arrangement in an inviscid fluid, conducted by Kármán, has disclosed the fact that for a particular ratio of spacing of vortices in each row to distance apart of rows, the arrangement is stable, and for that spacing only. Experimental investigation by Kármán and Rubach in the case of water verified that this is precisely the ratio established in fact by the eddies as produced. A much more exhaustive examination of the state of affairs in the wake has been conducted at the National Physical Laboratory, where, broadly speaking, it may be said that Kármán's results have been verified except in certain particulars due in the main to the fact that, while Kármán's vortices are infinitely thin filaments, the eddies formed in the air have finite cross-sections that increase as the eddy passes down the wake. Kármán's analysis is still not complete. What is still required is an investigation which will relate the final state of the wake as outlined above, with the state of affairs as the fluid leaves the body, that a connection may be established between the strength of the vorticity forming the vortices, and which, with the spacing of the two rows, determines the characteristics of the street on the one hand, and the velocity and dimensions of the body on the other. What is it, for example, that determines that only about 60 per cent. of the vorticity that leaves the edge of the cylinder goes towards forming the actual eddies, the remainder distributing itself generally throughout the fluid? Kármán's investigation nevertheless, although it does not tell the whole story, will stand as a very elegant and ingenious attempt to meet an extremely intricate problem. The present writer has recently attempted to extend the Kármán conception to three dimensions; here the problem is complicated by the possibility of rotational in addition to translational motions of the body. Moreover, because of the difficulties of technique, little experimental evidence regarding the nature of the wake in this case is available to act as a guide in forming the abstractions essential for a mathematical study. On general grounds, however, certain conclusions seem safe enough. For a non-rotating body—a circular disc, for example, in uniform rectilinear motion—one would expect that vortex rings would be shed in

succession from the edge. A long row of such rings have been investigated for a stable arrangement, but none has been found to occur. Provided the rings are spaced wider than a certain minimum amount, however, the system is practically stable. This appears to suggest that the stage of ring formation may be transitional to a more stable state. It would correspond in two dimensions to the symmetrical formation of vortex pairs behind a cylinder, a state of affairs that occurs only in the early history of eddying motion. The analogy to the alternate formation of vortices in a "street" of two rows presents slightly greater difficulty, but a good case can be made for supposing that this will consist of a spiral vortex unwinding itself from the edge of the three-dimensional body. If the body were elongated in one direction, the spiral elongated in that direction would practically consist of a set of parallel rectilinear filaments alternately spaced as in the Kármán arrangement. The steady motion and stability of a uniform spiral vortex of this type has now been investigated mathematically, and it is found that provided the pitch is greater than a certain critical amount the system will be stable. At the critical pitch the spiral changes its direction of rotation as a whole. If a and b be the radius of the spiral's cylinder and the standard dimension of the body, (w, v) and (Ω, V), the angular velocities and forward velocities of the spiral system as a whole and of the body, then if a relation can be established between $\frac{a\omega}{v}$ and $\frac{b\Omega}{V}$, two non-dimensional quantities for spiral system and body, it will now be possible to specify precisely the pitch of spiral that may be expected in the turbulent wake of a body in such a case. Theory in this respect has pushed ahead of experiment, and it seems desirable at the moment to stay further development until an experimental check on the results so far has been obtained. This is already being undertaken.

In the present survey an endeavour has been made to make plain the various lines of approach that have been made to a difficult subject, but one whose practical importance cannot be gainsaid. No reference has been made to a number of developments that in their day have had great vogue, as, for example, that involved in the Discontinuous Motion theory, for while they may have been important stepping-stones to a newer and a more realistic outlook, in themselves they can hardly be said to have contributed much of permanent value to a true Theory of Aerodynamics. H. LEVY.

GLEANINGS FAR AND NEAR.

671. Mental arithmetic among the mercantile castes of India.

"The Bania has a very distinctive caste character. From early boyhood he is trained to the keeping of accounts. . . . As an apprentice, he goes through a severe training in mental arithmetic, so as to enable him to make the most intricate calculations in his head. With this object a boy commits to memory a number of very elaborate tables. For whole numbers he learns by heart the units from one to ten multiplied as high as forty times, and the numbers from eleven to twenty multiplied to twenty times. There are also fractional tables, giving the results of multiplying $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{4}$, $1\frac{1}{2}$, $2\frac{1}{2}$ and $3\frac{1}{2}$ into units from one to one hundred; interest tables showing the interest due on any sum from one to one thousand rupees for one month, and for a quarter of a month at twelve per cent.; tables of the squares of all numbers from one to one hundred, and a set of technical rules for finding the price of a part from the price of the whole."—R. V. Russell, *The Tribes and Castes of the Central Provinces of India*, vol. ii. p. 128 [per Mr. F. Puryer White].

672. Note sur l'hypocycloïde à trois rebroussements.—Sommaire, *Bull. de Math. Spéc.* Nov. 1894.

THE DIFFERENTIATION OF a^x.

By T. P. Nunn, M.A., D.Sc., Litt.D.

Elsewhere I have shown how the study of compound interest leads to the conception of a "growth curve" (the exponential curve) from whose properties we can derive, first, the theory of logarithms, then the conception of e as the limit of $(1+1/n)^n$, and finally the differential coefficients of a^x and $\log_a x$. The method of differentiating a^x and of introducing e to be described here follows entirely different lines, and may be useful to students who have approached logarithms in the ordinary way through the theory of indices and have begun the study of the calculus before reading what is commonly called "higher algebra."

(1) Let $y=a^x$; then, by definition,

$$\frac{dy}{dx}=\underset{h\to 0}{\mathrm{L}}\frac{a^{x+h}-a^x}{h}$$

$$=a^x \,.\, \mathrm{L}\frac{a^h-1}{h}.$$

It is obvious that the value of $L(a^h-1)/h$ varies with the value of a. Let it, then, be assumed that there is a value of a, to be denoted by e, for which the value of the limit is unity. In the case of any other value of a let $a=e^u$; that is, let $u=\log_e a$. Then, in the first place, if $y=e^x$,

$$\frac{dy}{dx}=e^x,$$

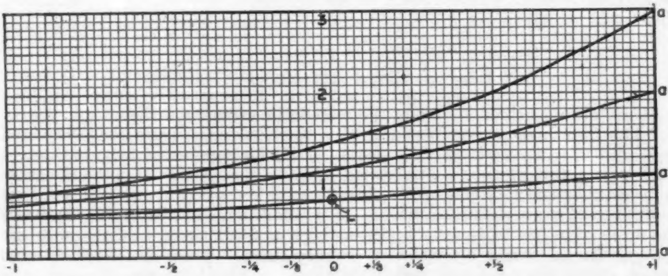
while, if $y=a^x=e^{ux}$, it follows further (by the rule for differentiating a function of a function of x) that

$$\frac{dy}{dx}=u\,.\,e^{ux}$$

$$=a^x\,.\,\log_e a.$$

Interchanging x and y, we can infer at once from this result that if

$$y=\log_a x,\quad dy/dx=1/(x\,.\,\log_e a).$$



(2) To complete the argument we must now justify the assumption upon which its validity depends. The simplest and most instructive way of doing this is to graph the values of the function $(a^x-1)/x$ for, say, 1, 2, 3, 4. With the help of a table of square roots and a table of reciprocals, numerical data sufficient for the task can quickly be accumulated; for it will be enough to table the values of the function in each case for $x=+1,\ +\frac{1}{2},\ +\frac{1}{4},\ +\frac{1}{8},\ -\frac{1}{8},\ -\frac{1}{4},\ -\frac{1}{2},\ -1$. For instance, if $a=3$, we have (from the square root table) $a^{\frac{1}{2}}=1{\cdot}732$; entering the same table with $1{\cdot}732$ we find that $a^{\frac{1}{4}}=1{\cdot}316$; and

a third entry with 1·316 gives $a^{\frac{1}{8}}=1{\cdot}147$. Turning with these numbers to the table of reciprocals we find that $a^{-\frac{1}{2}}={\cdot}577$, $a^{-\frac{1}{4}}={\cdot}760$, $a^{-\frac{1}{8}}={\cdot}872$. Inserting these values in the formula $y=(a^x-1)/x$ one can compute in a couple of minutes the values of the function for the decreasing series of values of x, namely: 2, 1·46, 1·26, 1·18, 1·02, ·96, ·85, ·67.

In this way the accompanying figure was drawn with a very small expenditure of time and labour. The limit of the function is, of course, indicated by the point where the corresponding curve cuts the y-axis. It is important to make the meaning of that statement clear to one's students. The graph labelled $a=2$ shows the value of $(2^x-1)/x$ for all values of x within its range with a single exception. That exception is zero, for which value of x the function has *no* value. It follows that the point marked L in the diagram, though it is on the curve labelled $a=2$, is not a point of the graph of the function: it marks a gap or "cut" in the graph, separating the parts which exhibit respectively the values of the function corresponding to positive and negative values of x. This, it should be understood, is what we mean when we say that the point L indicates the *limit* of the function as x approaches zero from above or from below.

It is evident from the graphs that the function has unity for its limit for a value of a somewhere between 2 and 3—apparently about 2·7.

(3) The evidence of the graphs points definitely to two conclusions:

(i) that e, as we have defined it, exists, and

(ii) that the limit of $(a^x-1)/x$, for any given value of a, lies, at least for small values of x, between the values of the function for $x=h$ and $x=-h$.* If we accept that evidence as sufficient, it is easy, with the help of an adequate table of common logarithms, to evaluate e. Suppose that $a=(1+\alpha)^p$, where α is a small positive fraction and p a large integer; and take $h=1/p$. Then, by (ii) above, the limit of $(a^x-1)/x$ lies, in this case, between the numbers

$$\frac{\{(1+\alpha)^p\}^{\frac{1}{p}}-1}{1/p} \quad \text{and} \quad \frac{\{(1+\alpha)^p\}^{-\frac{1}{p}}-1}{-1/p},$$

i.e. between $p\alpha$ and $p\alpha/(1+\alpha)$.

But, by paragraph (1),

$$\mathop{\mathrm{L}}_{x\to 0}\frac{a^x-1}{x}=\log_e a.$$

and since here $a=(1+\alpha)^p$ we have

$$p\alpha > \log_e(1+\alpha)^p > p\alpha/(1+\alpha).$$

Whence $$\alpha > \log_e(1+\alpha) > \alpha/(1+\alpha).$$

Now let $\log_{10}(1+\alpha)$ be picked out from a table of common logarithms, and suppose it to be divided by $\log_e(1+\alpha)$. Then the elementary formula $\log_p x=\log_q x \,.\, \log_p q$ shows that the quotient will be $\log_{10}e$; and it follows that $\log_{10}e$ must lie between

$$\frac{\log_{10}(1+\alpha)}{\alpha} \quad \text{and} \quad \frac{\log_{10}(1+\alpha)}{\alpha}(1+\alpha).$$

To apply this result, take $\alpha={\cdot}001$. Then, since $\log_{10}1{\cdot}001={\cdot}0004341$, we infer that $\log_{10}e$ lies between ·4341 and $\cdot4341\times1{\cdot}001={\cdot}4345$; that is, that e lies between 2·7177 and 2·7196. The mean of these numbers, namely 2·71865, cannot be far from its true value.

Seven-figure logarithms are not, however, capable of yielding a very satisfactory result in such a calculation as this, since, in the logarithms of numbers little greater than unity, the number of significant figures is too small. To obtain a closer estimate of e we must have recourse to a table in which the

* *I.e.* that the curve has no "kink" or point of inflexion where it crosses the y-axis.

logarithms are given to a much greater number of decimal places. Thomas Briggs (1556-1630), who first calculated logarithms to base 10, worked a certain number to the prodigious length of 61 places. From his table we learn that

$$\log_{10} 1{\cdot}000001 = {\cdot}000{,}000{,}434{,}294{,}264{,}756{,}155 \ldots .$$

In this case $a = {\cdot}000001$, and $\log_{10} e$ lies between $\cdot 43429426$ and $\cdot 4342947$. The seven-figure table gives the corresponding antilogarithms as $2{\cdot}71828$ and $2{\cdot}7182825$. We may conclude, accordingly, that the value of e to five places of decimals is $2{\cdot}71828$.

(4) It is not uncommonly believed that tables of logarithms to base 10 were originally computed by means of the expansions which are demonstrated in text-books on higher algebra and involve a knowledge of the value of e. If this idea were true, the calculation of the preceding paragraph would be a movement, so to speak, in a historical circle. But it is not true. Tables of common logarithms, which are the direct ancestors of the tables we use, were calculated by Briggs, by purely arithmetical methods, some time before the expansions were discovered. It follows that although, as a matter of history, e was not defined as the value of a for which the limit of $(a^x - 1)/x$ is unity, nor was first evaluated by the method of paragraph (3), yet both of these things might conceivably have happened. We may, I think, go further and admit that there is much to be said for introducing students—particularly, perhaps, engineering students—to the concept of e by means of the argument set out in this article. T. P. NUNN.

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673. . . . Now . . . that insidious . . . bowler . . . was sticking up the Clifton batsmen every over, to finish the match with 9 wickets for 20, and a victory by 66 runs. It was a magnificent achievement, but a liberal share of the credit must go to the captain, that indomitable leader, A. N. Whitehead, who made a cricketer of himself against his native quality for the sake of the school he had so finely led in football, and who is now writing so learnedly about science and mathematics, that it is to be feared that not one of his old team would understand a word of what he says. A great captain makes a great side, and a great captain is the central figure in the umpire's first cricket recollection.—Arthur Waugh in *A Cricket Eleven*, p. 10 [per Prof. E. H. Neville].

674. On a reproché à Roberval d'avoir dit d'une tragédie " qu'est ce que cela prouve ? " Mais il est de mode de prétendre que les mathématiciens n'ont pas toujours le sens très-juste.—M. L. Am. Sédillot, *Les Professeurs de Mathématiques . . . au Collège de France.* Rome, 1869, p. 125. [Cf. 325, 435, 617.]

675. *How we went down the tetradic steps.*—We went down one marble step underground where there was a resting . . . place ; then, turning to the left, we went down two other steps, where there was another resting place ; after that we came to three other steps, turning about, and met a third ; and the like at four steps, which we met afterwards. There quoth Panurge, Is it here ? How many steps have you told ? asked our magnificent lantern. One, two, three, four, answered Pantagruel. How much is that ? asked she. Ten, returned he. Multiply this, said she, according to the same Pythagorical tetrad. That is ten, twenty, thirty, forty, cried Pantagruel. How much is the whole ? said she. One hundred, answered Pantagruel. Add, continued she, the first cube—that's eight. At the end of that fatal number you'll find the temple gate ; and pray observe, this is the true psychogony of Plato, so celebrated by the Academics, yet so little understood ; one moiety of which consists of the unity of the two first numbers full of two square and two cubic numbers.—*Master Francis Rabelais, Five Books* . . ., V. c. 38, translation by Urquhart and Mottreux.

THE ORTHOCENTRE AND SOME PROPERTIES OF CONIC SECTIONS.

By C. Fox, M.A., D.Sc.

§1. The theory of the orthocentre is usually considered in conjunction with ordinary triangles only, that is to say, with triangles whose three vertices or sides are distinct. The treatment for the case when two or more sides or vertices coincide is generally overlooked, but it leads to many interesting results and applications. An exposition of some of these is the object of this note.

One result in particular may be new; it gives rise to the following simple geometrical construction for the radius of curvature ρ at a point P on a conic. *Let P' be the inverse of P with respect to the director circle. Let the perpendicular to PP' at P' cut the normal to the conic at P in the point R. Then $PR=\rho$.* This result is Theorem VIIIA below.

§2. A triangle ABC in which B and C coincide is known if we are given the positions of A and B, and also the direction of BC, while when A, B and C all coincide the triangle ABC is known if we are given the position of A and either the circumcircle of ABC, or the one ex-circle whose radius is not zero, both in magnitude and position. The orthocentre of such triangles can easily be found by various methods, one of which depends on the following well-known theorem.

Let H be the orthocentre of the triangle ABC and let the perpendicular from A to BC meet BC at D and the circumcircle of ABC at E. Then $ED=DH$ (Fig. 1).

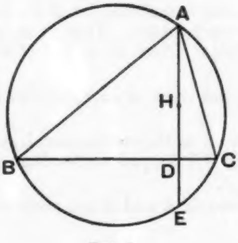

Fig. 1.

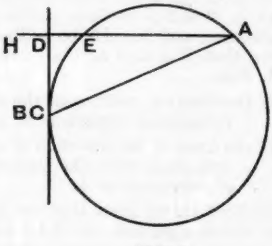

Fig. 2.

In other words, to find the orthocentre of a triangle ABC, we find the point where the perpendicular from A on to BC meets the circumcircle, and then find the reflection of this point in BC (Fig. 1).

If B and C coincide at B, then the chord BC becomes the tangent at B. Hence to find the orthocentre H of the triangle ABC in which B and C are coincident we proceed as follows:

I. *Draw the perpendicular from A to the tangent at B. If this perpendicular meets the circumcircle of ABC in E, then H is the reflection of E in the tangent at B* (Fig. 2).

If A, B and C coincide at A then BC becomes the tangent at A, while the perpendicular from A to BC becomes in the limit the diameter through A. Hence E is at the other end of the diameter through A. But the reflection of E in BC, *i.e.* the tangent at A, gives H, and so we have $EA=AO$. Hence the following result:

II. *The orthocentre H of a triangle ABC formed by three coincident points, the triangle being given by the position of A and the circumcircle of ABC is, situated on the diameter EA of this circumcircle, produced outwards so that* $EA = AH$ (Fig. 3).

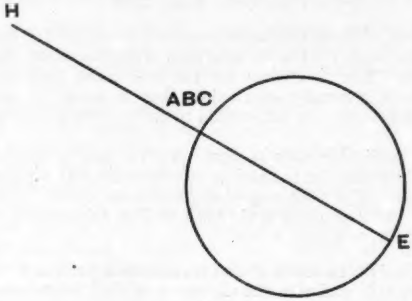

FIG. 3.

These results can also be obtained by other methods. For example, we may use the well-known fact that the circumcentre O, the centre of gravity G, and the orthocentre H of a triangle lie on the same straight line, the distances between them being connected by the relation

$$2OG = GH.$$

§ 3. Both I. and II. above have some interesting applications. As an illustration I give an application of II.

Let A, B and C be three points on a rectangular hyperbola, let H be their orthocentre, and let AE be a diameter of the circle ABC. Then it is well known that H is also on the rectangular hyperbola. Now let B and C tend to A, then

(i) the limiting position of the circle ABC is the circle of curvature to the rectangular hyperbola at A,

(ii) the limit of the direction of AE is the normal to the rectangular hyperbola at A, while the limit of the length of AE is equal to the diameter of curvature at A.

But from II. we know that the limiting positions of E and H are such that EAH is a straight line and $EA = AH$.

Hence we have the following result: *

III. *The diameter of curvature at a point A of a rectangular hyperbola is equal to the length cut off from the normal at A between the branches.*

§ 4. We can also make some interesting deductions for these limiting triangles if, instead of the circumcircle, we are given one of the ex-circles of the triangle. I shall deal only with the case in which all three vertices coincide.

If ABC is a triangle, then, in order that the radii of the circumcircle, the incircle and the three ex-circles shall *not* all tend to zero as A, B and C tend to coincidence, one of the angles, A say, must tend to π and the other two to zero. If R is the radius of the circumcircle and r_1 that of the ex-circle opposite A, then we know that

$$r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2},$$

and so, as $\hat{A} \to \pi$, $\hat{B} \to 0$, $\hat{C} \to 0$, the limit of $R/r_1 = \frac{1}{2}$.

* This result occurs amongst the miscellaneous examples in Russell's *Elementary Treatise on Pure Geometry* (Clarendon Press), p. 347, No. 169.

Again, since the limiting position of BC is tangential to the circumcircle of ABC, it follows that the two circles must touch at A in the limit, and since $\hat{A} \to \pi$, they must touch internally (Fig. 4).

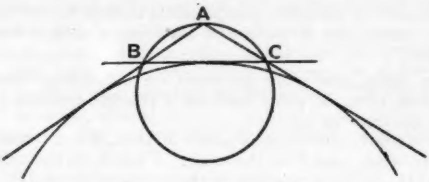

FIG. 4.

Hence we have the following result:

IV. *Given a triangle ABC, in which A, B and C coincide, and given its ex-circle of radius r, then its circumcircle is of radius $r/4$ and has internal contact with the ex-circle at A.*

Combining this with II. we then get:

V. *If AF is the diameter of the ex-circle of the triangle ABC, in which A, B and C all coincide, then the orthocentre H of ABC is situated at a point on FA produced so that $FA = 4AH$* (Fig. 5).

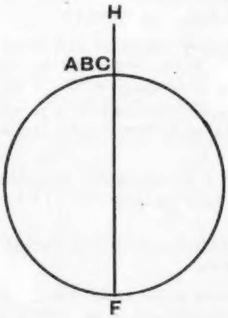

FIG. 5.

§5. We may use V. to obtain properties of circles of curvature to parabolae. Before we do so we assume the following theorem: *

VI. *Let ABC be a triangle touching any curve and let A, B and C all tend to some point P on the curve, then the limiting position of that ex-circle of ABC whose radius does not tend to zero is the circle of curvature at P.*

If the curve in VI. is a parabola, we know that the focus S lies on the circumcircle of ABC. Making A, B and C tend to coincidence with a point P on the parabola, it follows from IV. that the circle through S touching at P the tangent to the parabola at P has its radius equal to one-fourth that of the circle of curvature at P.

* This theorem is stated without proof by R. H. Fowler in *The Elementary Differentia Geometry of Curves* (Camb. Math. Tracts, (1929) No. 20), p. 31. It is not difficult to show that the radius of the limiting position of the ex-circle is given by $p + \frac{d^2p}{d\psi^2}$, where p is the perpendicular from the origin to a tangent and ψ the angle a tangent makes with the x-axis.

This is equivalent to the well-known result that the focal chord of curvature from a point P on a parabola is four times the join of P to the focus.

Again, it is known that the orthocentre of a triangle ABC touching a parabola is on the directrix. Hence from IV. we immediately deduce that:

VII. *The diameter of curvature in a parabola is equal to four times the intercept on the normal between the directrix and the point of intersection of the normal and the parabola.**

§ 6 Further deductions may be made on combining these results with Gaskin's theorem that the polar circle of a triangle circumscribing a conic is orthogonal to the director circle.

Let ABC be a triangle touching a conic and as A, B and C tend to coincidence let them tend to some point P on the conic. Then if ρ is the radius of curvature at P, and H is the limiting position of the orthocentre of ABC as A, B and C tend to P, we know from V. and VI. that H is on the normal at P produced, and also that $HP=\frac{1}{2}\rho$. But the limiting position of the polar circle to ABC is evidently the circle whose centre is H and whose radius is HP, and since this circle goes through P it is orthogonal to P. Also, by Gaskin's theorem, it is orthogonal to the director circle. Hence H is on the radical axis of P and the director circle.

This gives us the following result:

VIII. *Let ρ be the radius of curvature at a point P on a conic and let the radical axis of P and the director circle to the conic meet the normal at P to the conic at the point Q. Then $2PQ=\rho$.*

This result may also be stated as follows:

VIIIA. *Let ρ be the radius of curvature at a point P on a conic and let P' be the inverse of P with respect to the director circle. Let the perpendicular to PP' through P' cut the normal at P in the point R. Then $PR=\rho$.*

In the particular case when the conic is a parabola, VIII. reduces to VII. When the conic is a rectangular hyperbola, the director circle is the centre and VIII. reduces to

IX. *Let the normal at P to a rectangular hyperbola meet the curve again at N and let H lie between N and P so that $HP=\frac{1}{4}NP$. Then if C is the centre of the hyperbola, $OP=OC$.*

From III., HC is one-half the radius of curvature at P, and so we make the two following deductions:

X. *The locus of the centres of the rectangular hyperbolae which have a circle, (given both in magnitude and position), as the circle of curvature at the fixed point A, is a circle which touches the given circle externally at A and whose radius is half that of the given circle.*

XI. *The centres of rectangular hyperbolae which have a circle, (given both in magnitude and position,) as a circle of curvature, lie in the annulus bounded by the given circle and a concentric circle of twice its radius.* C. Fox.

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676. On a remarqué que chez certains poètes, chez Lucrèce, par exemple, l'enchaînement continu des idées est tel que l'on est aussi peu tenté d'en arracher un vers pour le citer seul, que de détacher une feuille d'un arbre ou un flot de la mer. L'ouvrage de M. Chasles [*Traité des sections coniques*] mérite la même louange et le même reproche.—J. Bertrand, *Journal des Savants*, 1866, p. 67.

677. I have never opened even the most elementary or childish book without earning something.—Leibniz.

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* This, of course, is well known, *v.* Russell, *loc. cit.* p. 347, No. 170.

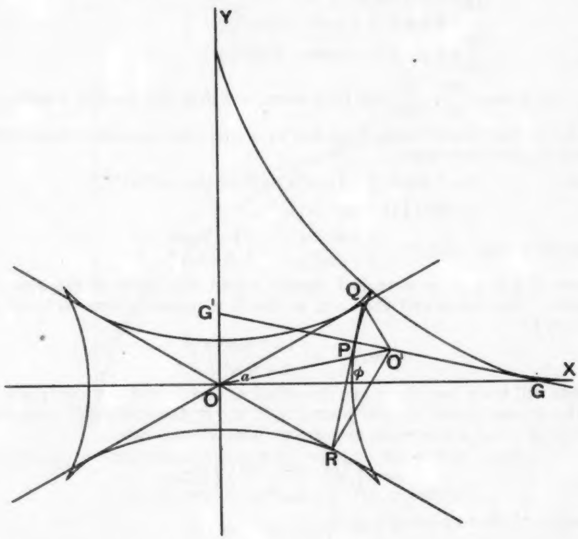
THE INVOLUTE OF THE ASTROID.

By H. V. Mallison, M.A.

If a rod of fixed length a slides with its ends on two fixed lines at right angles, its envelope is an astroid or hypocycloid of four cusps whose equation referred to the two fixed lines as axes is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

If the two lines are not at right angles but inclined at an angle 2α ($\alpha < 45°$), the envelope of the rod is a curve with four cusps in shape like an astroid elongated along one of the bisectors of the axes. This curve can be shown to be an involute of the astroid.

It is convenient to take the bisectors of the two lines as axes.



The point of contact of the rod with its envelope can be found as follows. If QR is the position of the rod and QO', RO' perpendicular to OQ, OR meet at O', then O' is the instantaneous centre of rotation for the rod and P, the foot of the perpendicular from O' on QR, is the point of contact of QR with its envelope.

Let QR make an angle ϕ with OX.

Then as $OQO'R$ is cyclic $\angle O'OR=\angle O'QR=90°-\angle OQR=90°-(\phi-\alpha)$. Thus $\angle O'OX=90°-(\phi-\alpha)-\alpha=90°-\phi$.

Also $\angle PGO=90°-\phi$.

Thus it follows that if PG, the normal at P to the locus of P, cuts OX, OY at G, G', O' is the mid-point of GG'. Therefore

$$GG'=2O'G=2OO'=2\,.\,a/\sin 2\alpha=\text{constant}=2b, \text{ say.}$$

Hence the envelope of GG' is the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(2b)^{\frac{2}{3}}$, which is the evolute of the locus of P.

The parametric form of the locus can be easily found.

$$\begin{aligned} x &= \text{sum of projections on the } x\text{-axis of } OQ,\ QO',\ O'P \\ &= OQ \cos\alpha + QO' \sin\alpha - O'P \sin\phi \\ &= b\sin(\alpha+\phi)\cos\alpha + b\cos(\alpha+\phi)\sin\alpha + b\cos(\alpha+\phi) \\ &\qquad \cos(\alpha-\phi)\sin\phi \\ &= b\sin\phi + \tfrac{1}{2}b\sin\phi\,(\cos 2\phi + \cos 2\alpha). \end{aligned}$$

Hence $x = b\sin\phi\,(\cos^2\phi + \cos^2\alpha)$.

Similarly $y = b\cos\phi\,(\sin^2\phi + \sin^2\alpha)$.

These give the parametric equations of the curve.

$$\begin{aligned} \frac{dx}{d\phi} &= b\cos\phi\,(\cos^2\phi + \cos^2\alpha) - 2b\sin^2\phi\cos\phi \\ &= b\cos\phi\,(1 + \cos^2\alpha - 3\sin^2\phi), \\ \frac{dy}{d\phi} &= b\sin\phi\,(1 + \cos^2\alpha - 3\sin^2\phi). \end{aligned}$$

At a cusp $\dfrac{dx}{d\phi}$, $\dfrac{dy}{d\phi}$ are both zero, and thus $\sin^2\phi = \frac{1}{3}(1+\cos^2\alpha)$.

As the radius of curvature vanishes at a cusp, the astroid-evolute will pass through each of the cusps.

Here $x = b\sin\phi\,(\frac{2}{3} + \frac{2}{3}\cos^2\alpha) = 2b\{\frac{1}{3}(1+\cos^2\alpha)\}^{\frac{3}{2}}$,

$y = 2b\{\frac{1}{3}(1+\sin^2\alpha)\}^{\frac{3}{2}}$.

Thus at a cusp $\tan\theta = \left(\dfrac{1+\sin^2\alpha}{1+\cos^2\alpha}\right)^{\frac{3}{2}} = \left(\dfrac{1+2\tan^2\alpha}{2+\tan^2\alpha}\right)^{\frac{3}{2}}$.

Hence if a thread is stretched tightly round the inside of the astroid and fastened at the cusps and then cut at the four points whose vectorial angles are given by

$$\tan\theta = \pm\left(\frac{1+2\tan^2\alpha}{2+\tan^2\alpha}\right)^{\frac{3}{2}},$$

the ends will trace out the curve described as the thread is unwrapped.

If the thread is cut at the points $\pm\theta$, $\pi\pm\theta$, the ends will trace out the envelope of a rod whose ends are on two lines at an angle

$$\cos^{-1}\left\{\frac{3(1-\tan^{\frac{2}{3}}\theta)}{1+\tan^{\frac{2}{3}}\theta}\right\}\ (=2\alpha),$$

the length of the rod being $b\sin 2\alpha$.

Also
$$\begin{aligned} \pm\frac{ds}{d\phi} &= b\,(1+\cos^2\alpha - 3\sin^2\phi) \\ &= \tfrac{1}{2}b\,(\cos 2\alpha + 3\cos 2\phi), \end{aligned}$$

so that
$$s = \tfrac{1}{2}b\,(\phi\cos 2\alpha + \tfrac{3}{2}\sin 2\phi),$$

where s is measured from the point at which the curve meets the y-axis, where $\phi = 0$.

From this it may be deduced that the total length of the curve

$$= 2b\,(3 + \pi\cos 2\alpha).$$

The area of the part in the positive quadrant is $\int y\,dx$ from $\phi = 0$ to $\phi = \dfrac{\pi}{2}$, as the part of the area from $\phi = \phi_1$ at the cusp to $\phi = \dfrac{\pi}{2}$ is negative. The total area will be found to be $\frac{1}{8}b^2\pi(3 - 2\cos^2 2\alpha)$.

The cartesian equation of the curve is a long expression of little interest, but it will be seen that the curve, like the astroid, is of the sixth degree.

I am indebted to Mr. J. P. Wiles, M.A., for his work on the above problem and also for a model of the figure. H. V. Mallison.

MATHEMATICAL NOTES.

933. [L[1]. 9.] *A simple expression for the area of a segment of a parabola.*

Let the join of $P(x_1, y_1)$, $Q(x_2, y_2)$ on $y^2=4ax$ be the polar of $T(x'. y')$.

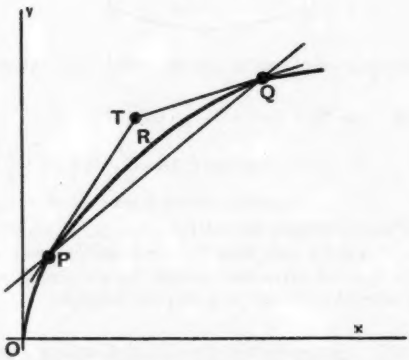
Then area of segment $PRQ = \int_P^Q 2a^{\frac{1}{2}}x^{\frac{1}{2}}\,.\,dx\,.\,-\frac{(y_2+y_1)(x_2-x_1)}{2}$

$$=\frac{2}{3}(x_2y_2-x_1y_1)-\tfrac{1}{2}(y_2+y_1)(x_2-x_1)$$

$$=\frac{1}{6}[x_2y_2-x_1y_1-3x_2y_1+3x_1y_2]$$

$$=\frac{1}{24a}[y_2^3-y_1^3-3y_1y_2^2+3y_1^2y_2]$$

$$=\frac{(y_2-y_1)^3}{24a}.$$



$(x_1, y_1)(x_2, y_2)$ satisfy the equations $yy'=2a(x+x')$ and $y^2=4ax$;

$$\therefore\ y_1, y_2 \text{ are the roots of } \frac{y^2}{2}-yy'+2ax'=0;$$

$$\therefore\ y_2+y_1=2y',\ y_2y_1=4ax';$$

$$\therefore\ y_2-y_1=2\sqrt{y'^2-4ax'};$$

$$\therefore\ \text{area}=\frac{[y'^2-4ax']^{\frac{3}{2}}}{3a}.$$

Ex. *To find the area cut off from* $y^2=8x$ *by* $y=x-1$.

The pole of $y=x-1$ w.r.t. $y^2=8x$ is found to be $(-1, 4)$;

$$\therefore\ \text{area}=\frac{[y'^2-4ax']^{\frac{3}{2}}}{3a}=\frac{[4^2-4(2)(-1)]^{\frac{3}{2}}}{3(2)}=8\sqrt{6}.$$

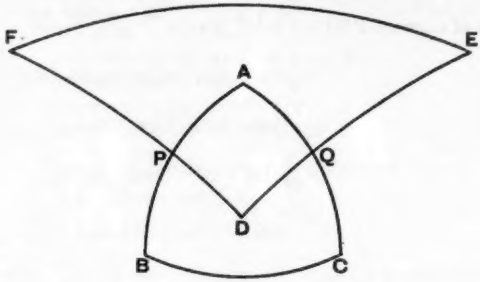
H.M. Dockyard School, Devonport. V. Naylor.

934. [K[1]. 20. f.] *The Geometrical Interpretation of Cagnoli's Equation*:

$$\sin b\sin c+\cos b\cos c\cos A=\sin B\sin C-\cos B\cos C\cos a.$$

If we apply the polar transformation to the above equation, we do not obtain a new one, and so it expresses a relation which is common to a triangle and its polar triangle.

Let ABC be a spherical triangle. On BA take BP=a quadrant, and on CA take CQ=a quadrant. Through P draw a great circle at right angles to BA, and similarly through Q. Let these meet at D. Then DP is the polar



of B, and QD is the polar of C, and so D is the pole of BC. Complete the polar triangle DEF.

Then in $\triangle APQ$, $$\cos PQ = \cos\left(AB - \frac{\pi}{2}\right)\cos\left(AC - \frac{\pi}{2}\right) + \sin\left(AB - \frac{\pi}{2}\right)\sin\left(AC - \frac{\pi}{2}\right)\cos A$$
$$= \sin b \sin c + \cos b \cos c \cos A,$$
and considering PQ as belonging to $\triangle DPQ$,
$$\cos PQ = \sin B \sin C - \cos B \cos C \cos a,$$
so each side of the equation gives the cosine of the arc joining the intersections of corresponding sides of a triangle and its polar triangle.

D. PEDOE.

935. [B. 1. b.; O[1].] *An Application of the Multiplication of Determinants to the Geometry of Surfaces.*

The well-known method for the multiplication of two determinants of the same order may be applied with advantage to several theorems in the Geometry of Surfaces. By multiplying by an expression equal to unity (which is the quotient of a determinant and its value when expanded), much inelegant and tedious manipulation used in standard works to derive one determinant expression from another may be eliminated. A similar method of working is implied by Prof. H. Hilton in his paper "on the projection of one surface on another" [*Messenger of Mathematics*, vol. lvii. p. 84 (1927)], but he does not seem to have applied it to the cases dealt with below.

The six fundamental quantities of a surface referred to parameters u, v are defined as
$$E = \Sigma x_1^2,\ F = \Sigma x_1 x_2,\ G = \Sigma x_2^2,\ L = \Sigma X x_{11},\ M = \Sigma X x_{12},\ N = \Sigma X x_{22};$$
where X, Y, Z are the direction-cosines of the normal to the surface at the point (x, y, z), and the suffixes $_1$ and $_2$ denote partial differentiation with respect to u and v respectively.

Summarising some of the elementary results, we have:
$$X = (y_1 z_2 - y_2 z_1)/V,\ Y = (z_1 x_2 - z_2 x_1)/V,\ Z = (x_1 y_2 - x_2 y_1)/V;$$
$$\Sigma X^2 = 1,\ \Sigma X X_1 = \Sigma X X_2 = \Sigma X x_1 = \Sigma X x_2 = 0;$$
$$L = -\Sigma X_1 x_1,\ M = -\Sigma X_1 x_2 = -\Sigma X_2 x_1,\ N = -\Sigma X_2 x_2;$$
where for brevity we write $EG - F^2 = V^2$.

Hence $\begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \Sigma X(y_1 z_2 - y_2 z_1) = \Sigma X \,.\, XV = V.$ ....(1)

1. We may now evaluate the determinant

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} \;(=\Delta \text{ say}),$$

for which Prof. A. R. Forsyth (*Diff. Geom.* p. 40) indicates the use of equations such as those giving X_1 in terms of x_1 and x_2.

From (1), we have

$$\Delta = \frac{1}{V}\begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \cdot \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$= \frac{1}{V}\begin{vmatrix} \Sigma X^2 & \Sigma X X_1 & \Sigma X X_2 \\ \Sigma x_1 X & \Sigma x_1 X_1 & \Sigma x_1 X_2 \\ \Sigma x_2 X & \Sigma x_2 X_1 & \Sigma x_2 X_2 \end{vmatrix}$$

$$= \frac{1}{V}\begin{vmatrix} 1 & 0 & 0 \\ 0 & -L & -M \\ 0 & -M & -N \end{vmatrix} = \frac{LN - M^2}{V}.$$

2. In his *Differential Geometry*, p. 84, Prof. C. E. Weatherburn derives the following expression for the torsion of an asymptotic line on a surface:

$$\tau = \begin{vmatrix} X & Y & Z \\ X' & Y' & Z' \\ x' & y' & z' \end{vmatrix},$$

where $'$ denotes differentiation with respect to the arc-length s of the asymptotic line.

Multiplying by the unit expression

$$\frac{1}{V}\begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

we obtain

$$\tau = \frac{1}{V}\begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \cdot \begin{vmatrix} X & Y & Z \\ X' & Y' & Z' \\ x' & y' & z' \end{vmatrix}$$

$$= \frac{1}{V}\begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \cdot \begin{vmatrix} X & Y & Z \\ X_1 u' + X_2 v' & Y_1 u' + Y_2 v' & Z_1 u' + Z_2 v' \\ x_1 u' + x_2 v' & y_1 u' + y_2 v' & z_1 u' + z_2 v' \end{vmatrix}$$

$$= \frac{1}{V}\begin{vmatrix} \Sigma X^2 & \Sigma(X X_1 u' + X X_2 v') & \Sigma(X x_1 u' + X x_2 v') \\ \Sigma x_1 X & \Sigma(x_1 X_1 u' + x_1 X_2 v') & \Sigma(x_1^2 u' + x_1 x_2 v') \\ \Sigma x_2 X & \Sigma(x_2 X_1 u' + x_2 X_2 v') & \Sigma(x_2 x_1 u' + x_2^2 v') \end{vmatrix}$$

$$=\frac{1}{V}\begin{vmatrix} 1 & 0 & 0 \\ 0 & -Lu'-Mv' & Eu'+Fv' \\ 0 & -Mu'-Nv' & Fu'+Gv' \end{vmatrix}$$

$$=\frac{1}{V}\begin{vmatrix} Eu'+Fv' & Fu'+Gv' \\ Lu'+Mv' & Mu'+Nv' \end{vmatrix},$$

which agrees with Weatherburn's result.

From his definition of lines of curvature (*Diff. Geom.* p. 66), the normals to the surface at "consecutive" points of the curves intersect (neglecting distances of order higher than the first). Hence their equation is given by

$$\begin{vmatrix} X & Y & Z \\ dX & dY & dZ \\ dx & dy & dz \end{vmatrix}=0.$$

By the preceding work this leads to

$$\begin{vmatrix} Edu+Fdv & Fdu+Gdv \\ Ldu+Mdv & Mdu+Ndv \end{vmatrix}=0.$$

3. We may now simplify Weatherburn's method of determining the shortest distance between two "consecutive" lines of a congruence (*Diff. Geom.* p. 192), and also their mutual moment.

In what follows, X, Y, Z denote the direction-cosines of the line of the congruence,

$$a=\Sigma x_1X_1,\ b=\Sigma x_2X_1,\ b'=\Sigma x_1X_2,\ c=\Sigma x_2X_2,\ e=\Sigma X_1^2,\ f=\Sigma X_1X_2,\ g=\Sigma X_2^2.$$

As $$\Sigma X^2=1,\quad \Sigma XX_1=0,\quad \Sigma XX_2=0.$$

Hence $$\frac{X}{Y_1Z_2-Y_2Z_1}=\frac{Y}{Z_1X_2-Z_2X_1}=\frac{Z}{X_1Y_2-X_2Y_1}=\frac{1}{\{\Sigma(Y_1Z_2-Y_2Z_1)^2\}^{\frac{1}{2}}}=\frac{1}{h},$$

where $h^2=eg-f^2$.

Further $$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}=\Sigma X(Y_1Z_2-Y_2Z_1)=\Sigma X\,.\,Xh=h. \quad\text{...............(2)}$$

Weatherburn obtains the mutual moment in the form $D\,.\,ds^2$, where

$$D=-\begin{vmatrix} X & Y & Z \\ X' & Y' & Z' \\ x' & y' & z' \end{vmatrix}.$$

From (2), $$D\,.\,ds^2=-\frac{ds^2}{h}\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}\,.\,\begin{vmatrix} X & Y & Z \\ X' & Y' & Z' \\ x' & y' & z' \end{vmatrix}$$

$$=-\frac{1}{h}\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}\,.\,\begin{vmatrix} X & Y & Z \\ X_1du+X_2dv & Y_1du+Y_2dv & Z_1du+Z_2dv \\ x_1du+x_2dv & y_1du+y_2dv & z_1du+z_2dv \end{vmatrix}$$

$$= -\frac{1}{h}\begin{vmatrix} 1 & 0 & \Sigma(Xx_1du+Xx_2dv) \\ 0 & edu+fdv & adu+bdv \\ 0 & fdu+gdv & b'du+cdv \end{vmatrix}$$

$$= \frac{1}{h}\begin{vmatrix} a\,du+b\,dv & b'du+c\,dv \\ e\,du+f\,dv & f\,du+g\,dv \end{vmatrix}.$$

The parameter of distribution of the congruence follows immediately.

University College, Nottingham. L. E. PRIOR.

936. [M^2. 1. a.] *Quadrically Associated Points.*

Hesse's work on the bitangents to a plane quartic (*Crelle*, Bd. 49) implies the existence of some theorems on associated points which do not seem to have been noticed before: Hesse obtains his results without appealing to them.

Eight points 12345678 are "associated" if they are the intersection of three quadrics. The twisted cubic through any six of them meets the join of the remaining two in two points P, Q. We then have

(i) The points P, Q on the four lines of types 12, 23, 34, 41 and 12, 34, 56, 78 also form sets of associated points. These sets correspond to the "syzygetic fours"; the result is easily proved by the theorem on self-polar tetrads inscribed in a twisted cubic (Baker, *Principles of Geometry*, iii. p. 147).

(ii) The points P, Q on the four sets of types 12, 13, 14, 15, 16, 78; 12, 13, 14, 56, 57, 58; 12, 23, 34, 45, 56, 61; and 12, 23, 34, 42, 45, 67 lie on a quadric. To these correspond the hexads of bitangents whose points of contact lie on a cubic; at present, however, I see no way of establishing this property from first principles. L. ROTH.

Imperial College, S.W. 7, 10th May, 1927.

937. [O^1. 5.] *On the Divergence and Circulation Theorems for a Surface.*

The divergence and circulation theorems, recently discovered by the writer for point-functions on a given surface, were proved independently.* The latter may, however, be neatly deduced from the former as follows. Let $\mathbf{F}$ be a vector function on the surface and $\mathbf{n}$ the unit normal. Apply the divergence theorem to the vector $\mathbf{F}\times\mathbf{n}$. Then for a closed curve C on the surface, if ds is the length of an element of arc, $\mathbf{t}$ the unit tangent, and $\mathbf{m}$ the unit vector tangential to the surface and perpendicular to $\mathbf{t}$, drawn outward, we have

$$\iint \operatorname{div} \mathbf{F}\times\mathbf{n}\,dS = \int_0 \mathbf{F}\times\mathbf{n}\,.\,\mathbf{m}\,ds, \quad\text{.............................(1)}$$

the line integral being taken round the curve C, and the surface integral over the region enclosed. Now

$$\mathbf{F}\times\mathbf{n}\,.\,\mathbf{m} = \mathbf{F}\,.\,\mathbf{n}\times\mathbf{m} = \mathbf{F}\,.\,\mathbf{t}$$

and
$$\operatorname{div} \mathbf{F}\times\mathbf{n} = \mathbf{n}\,.\,\operatorname{curl}\mathbf{F},$$

since curl $\mathbf{n}$ is zero. Hence we may write (1)

$$\iint \mathbf{n}\,.\,\operatorname{curl}\mathbf{F}\,dS = \int_0 \mathbf{F}\,.\,\mathbf{t}\,ds = \int_0 \mathbf{F}\,.\,d\mathbf{r}, \quad\text{.........................(2)}$$

which is the circulation theorem for the function $\mathbf{F}$ and the curve C.

The function curl $\mathbf{F}$ in the above theorem is a two-parametric invariant for the given surface. If, however, $\mathbf{F}$ is a point-function in space, the ordinary three-parametric invariant curl $\mathbf{F}$ is different from the preceding: but the two vectors have the same resolved part in the direction † of $\mathbf{n}$, that is to say,

* *Quarterly Journal of Math.* vol. 50, pp. 253-258 (1925), or the author's *Differential Geometry*, Arts. 122-124 (C.U.P. Jan. 1927). The differential invariants here employed are all two-parametric.

† *Proc. Roy. Soc. Edin.* vol. 46, part 2, p. 200 (1926).

n . curl **F** has the same value for both functions. In the case of the three-parametric invariant the equation (2) expresses Stokes's theorem. Hence Stokes's circulation theorem may be deduced from the writer's divergence theorem for a surface. C. E. WEATHERBURN.

Canterbury College, N.Z.

938. [K. 2. a.] *The Generalisation of the Steiner Envelope* [see *Gazette*, vol. xiv., p. 142: Note 898].

The generalisation of the Steiner Envelope stated by Mr. L. J. Rogers, viz. "The perpendiculars are drawn to the sides of a triangle at the points where a transversal cuts them, forming a similar triangle. If the magnitude of the triangle is constant, the envelope of the transversal will be a three cusped *hypotrochoid*" requires correction. The envelope is not a hypotrochoid but a pair of tricusp hypocycloids which coalesce with the Steiner tricusp when the magnitude of the triangle becomes zero. This and some other theorems relating to the transversal are contained in *Journal of the Indian Mathematical Society*, vol. xviii. No. 1. G. A. SIMIVASAN.

Presidency College, Madras.

939. [V. 1. a. ξ, η]. *Economy in the Elements of Mathematics.*

Perhaps there are various fundamental points of view from which the relations of constructs in space, that is, geometry, may be studied. All must be interesting when treated on rational lines. You have been getting on to this philosophy of mathematics as a refuge from the tedious and interminable detail, refined without end, which seems to be inseparable from elementary work. Examination "cuts" are blamed; but they are often works of concise art—ask any lawyer of the past generation whose wits were sharpened on the mathematical tripos—far removed from the elementary research of which one can complain. By all means record it in print for future ages; but keep the schoolboy and undergraduate away from it until he can exercise a free choice. If it is over-developed, the intrinsic study of the elements, as decided from the points of view of the classical masters, must go to the wall.

Geometric facts, even isolated cuts, have often a charm of their own. But what is to be said of the *juggernaut* of algebra, including what is called trigonometry and some kinds of mechanics? Cannot something be done there towards saving of time and sense?

In the line of your present geometric inquiries may one propound the following questions?

(i) Any configurational proposition about lines in space may be projected from an eye situated at any point into its background, thus involving a proposition about great circles on a sphere, or ultimately about lines in a plane?

(ii) There are configurational propositions on the plane or surface which have no analogues in three dimensions, *e.g.* the one so-called of Pappus, because in fact the lines in space do not usually meet. So plane geometry must be more than a geometry derivable from shadows of spatial configurations.

(iii) It appears that it encircles *only one* additional proposition or principle for a complete foundation, *e.g.* this theorem of Pappus, which suggests a question, perhaps too indefinite: What proportion of configurational theorems in a plane are projections of intuitively obvious configurations in space without this extra foundation?

(iv) Is the Cayleyan "absolute" quadric or conic an essential element in projective geometry, widened so as to include metrics, or only a limiting form thereof represented for the conic by two points? INQUIRER.

940. [P^1. 1. b.] *A Problem in Geometrical Construction.*

The following problem was proposed in a Civil Service examination a few years ago:

"A parallelogram is drawn on paper. Construct the square of which it is the orthogonal projection."

Perhaps the subjoined solution "by conics" may interest readers of the *Gazette*:

Inscribe an ellipse in the parallelogram so that its centre is the centre of figure and it has a pair of conjugate diameters parallel to sides of the parallelogram. When a circle is orthogonally projected into this ellipse the diameter of this circle is equal to the major axis of the ellipse. The problem thus resolves itself into the well-known construction for the axes of an ellipse, given a pair of conjugate diameters in magnitude and direction; viz. if CP and CD are the conjugate semi-diameters, draw PF perpendicular to CD and set off $PL=PL'=CD$ on this line. CL and CL' are the sum and difference of the semiaxes of the ellipse, and the directions of the axes are the bisectors of the angle LCL'. The major axis of the ellipse is, of course, the side of the required square.

E. H. SMART.

941. [V. 1. a.] *Why all this fuss?*

The fuss is, of course, about triangles. Euclid shares with other writers of antiquity the honour of a place in our curricula as a "cultural" force. It is not on account of its results that geometry is taught as it is taught. The results could be reached equally well by experimental drawing. The teaching is justified rather as a training in clear, logical thinking. It is also justified, in another sense, by the prestige of having nearly half of all the time devoted to mathematics. We are most of us so impressed by this as to feel that, after all, there must be something in it.

The amount of time and labour given to geometrical theory, by tradition, is no evidence of its value. Consider the enormous importance of incantation and sacred ceremonies among primitive peoples. Such rituals are the traditional accompaniment of the sowing and harvesting of crops. Modern methods of "preparation and propitiation" lay more emphasis on soil-chemistry and seed-testing. It is not implied that school-geometry partakes of this ceremonial character; but any ancient discipline lies open to the encroachments of the sanctifying paralysis of age. Further, an acquaintance with the history of the subject reinforces this attitude of distrust. Geometry was recovered for Europe from the Arabic-Jewish scholars, who in their turn had learned it from the text-books of the Alexandrian schools. It came, also, more directly, by way of the Byzantine lecturers and their copies of the Greek originals. Algebra, on the other hand, is a modern study. In spite of its practice among the Arabs, algebra as we know it is a new thing. Its present form and power are due to the invention by Vieta of a scientific symbolism in 1591.

Geometry, then, gets more reverence, perhaps, simply because of its hoary antiquity.

There is a second pillar supporting this amazing structure. It is argued that the discipline produces minds capable of logical reasoning and critical thinking. The same claim has been made on behalf of diverse practices, such as writing Latin verse and learning the Shorter Catechism. Experience hardly justifies these claims. Indeed, recent experimental studies have thrown doubt upon this transfer of training. It is found that habits of neatness, for example, formed in one subject are not carried over into another, unless special attention has been drawn to the habits as ideals in themselves. This effectually shatters the other pillar of the faith.

But the attack may be carried further. Admitting that a good training in logical thought is excellent and necessary, we may insist that it should be the form of logic which approximates most closely to the logic of everyday life and workaday experience. Now, everyday logic is concerned with how things change together, that is, with one thing as a function of another. For instance,

dividends are quoted as percentages, taxes levied as so much in the pound; there is a relative drop in overhead costs with the expansion of business; unemployment figures vary with the bank rate and the world harvest, families change in size as one passes from one stratum of incomes to another; there are piece-work rates, salary increments and so forth. Popular articles on these subjects are frequently illustrated by graphs—which are simply pictures of functions. And the logic or science of functions is modern algebra.

Workaday logic makes wide uses of this logic. There is scarcely a technical industry to-day without its characteristic group of functions written in a special shorthand, also part of algebra. The electrician, the works chemist, the builder, the engineer, all have their tables of constants and their lists of formulas for conductivity, reaction velocities, elasticity, winding moments, and what not. These, and their fellows too, make use of the formula-pictures, graphs.

In the research laboratory exact results are weighed and conned and checked until formulas are found which cover them and make calculation and further advance possible. From the simple, well-known $PV=RT$ of the gas law to such interesting statements as $-\Gamma=\frac{C}{R}\cdot\frac{ds}{dC}$; from the familiar, electrical $C=\frac{E}{R}$ to $q=e^{-\frac{R}{2L}t}A\cos(pt-B)$; from mere averages to the complex reckonings of statistics, algebra with all its varied symbols and its very powerful engines of analysis—the calculus, etc.—is indispensable to both exact calculation and intelligible report.

Yet this important branch of logic is overshadowed by a discipline unrelated to modern thought and burdened with an unhealthy purity-complex. For what could be more ridiculous than to prove by " purely geometrical methods " that " the square on a line is equal to the sum of the squares upon two segments, together with twice the rectangle contained by the segments " ? As a game in cutting up paper, this might be amusing to a young child; as an exercise in a sort of stylistic virtuosity it might interest a connoisseur of the harmlessly useless; but as a part of school mathematics it is a wicked waste of time. Geometry is a logic of form, but the essence of modern mathematics is that it is a logic of function. To-day Euclidean, or school, geometry, is merely an unimportant member of a rich and mighty assemblage of geometries.

D. N. L.

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678. Vn Frippier veut vendre vne robbe 64 liures à vn Aduocat indecrotable, qui n'auoit pas beaucoup de pecune. Toutefois voyant sa robbe assez deschirée, et bien crottée, et que celle qu'il marchandoit lui sembloit belle et bien faicte, dict au Frippier, qu'il en donnerait 428.939.858 liures, dix sols quatre deniers, (qui valent 102.945.566.047.324 deniers) en rabattant tousiours de la tierce partie iusques à 30 fois, le Frippier accepta le marché, et donna la robbe à l'Aduocat, qui la receut d'vne gayeté de coeur, et amena le dit Frippier en son logis et firent conte ensemble, et l'Aduocat se trouua redeuable au Frippier d'vn denier, lequel il prend en sa bourse, et le donne au Frippier. Et par ce moyen Monsieur l'Aduocat se trouua bien paré et à peu de frais, par le moyen de la subtilité d'une reigle d'Arithmatique.—*Methodiques Institutions de la vraie et parfaite arithmetique de Jacques Chauvet.* Reveue, corrigée, et amplifiée d'exemples Geometriques. *Par P. Taillefer . . .* Paris, M.CD.XV.

679. The father of a new student when brining him to the University, after calling to see the Professor [Sir William Thomson] drew his assistant to one side and besought him to tell him what his son must do that he might stand well with the Professor. " You want your son to stand weel with the Professor ? " " Yes." " Weel, then, he must just have a guid bellyful o' mathematics."—Silvanus Thompson, *Life of Lord Kelvin* (1910), p. 420.

REVIEWS.

Calculus. By H. B. Phillips. Pp. 349. 15s. 1928. (Wiley & Sons.)

In the preface to this book the author asserts that he has "endeavoured to emphasize those principles that are found most useful in applications to science and engineering." In pursuit of this end, he touches very lightly on the fundamental concepts of the Calculus. Thus, for example, the treatment of limiting processes is so slight as to be, in more than one particular, definitely misleading to a beginner.

The book then proceeds on the customary lines, ending with chapters devoted to Taylor's Theorem and Differential Equations. The elementary parts of the theory of double and triple integrals, however, are given more space than is usual in a work of this type, and an attempt is made to treat the differential and its applications in a rigorous fashion. Though this attempt is, perhaps, not completely successful, yet the reader will be materially assisted for any future efforts to reconcile the precise "differential" of the pure mathematician with the "little bit" of the physicist.

The science or engineering student who believes that it is possible to acquire a working knowledge of the Calculus without troubling about the mathematical principles of the subject, will find this book to his taste. But it is not suitable for a mathematical student. T. A. A. B.

An Elementary Treatise on Differential Equations and their Applications. By H. T. H. Piaggio. Pp. xviii + 256 + xxvii. 12s. net. 1928. (G. Bell & Sons.)

It is a pleasure to welcome the second edition of Professor Piaggio's admirable text-book. A revision of the text of the first edition has been made, and a new chapter, some forty pages long, on Miscellaneous Methods, has been added.

The merits of the earlier edition have been so universally recognised that further praise would be superfluous; it is only necessary to consider the additional chapter. This is divided into six parts, the second of which is concerned with Riccati's Equation, a subject which was dismissed by a few exercises in the first edition. The remaining five parts are in the nature of addenda to preceding chapters and deal with

(1) the theory of singular solutions,
(3) Mayer's method for integrating $Pdx + Qdy + Rdz = 0$,
(4) solutions in series,
(5) the Equation of Wave Motion,
(6) numerical solutions.

Here (3) and (5) call for little comment. In (1) the difficulties of the problem are clearly stated, and the interesting conception of exceptional loci as boundary curves between the regions is put forward. The theory of solutions in series of Linear Differential Equations of the second order, given in Chap. X., is in (4) expanded and analysed; the notion of a regular integral—that is, a solution in Frobenius' form—is here made precise.

Finally, (6) deals partly with Remes' analysis of Professor Piaggio's own method of numerical solution, but mainly with a method due to J. C. Adams * not mentioned in the first edition. The author suggests that this is perhaps the best method of numerical approximation, and this opinion is said to be shared by Professor Whittaker as a result of tests held in the Edinburgh Mathematical Laboratory.

The usefulness of the book is indeed enhanced by these additions, which, however, might well have been incorporated in the body of the work. That they were not, is probably due to a desire to allow references to the first edition to remain applicable to the second. If the nice balance of the first edition is a little disturbed, the defect is only trivial, but then a reviewer must find some point which he can criticize adversely, and Professor Piaggio makes the task difficult. T. A. A. B.

* But surely a member of St. John's College should not suggest that Adams deduced the existence of Neptune from the perturbations of *Saturn*!

The Theory of Determinants, Matrices and Invariants. By H. W. Turnbull. Pp. xvi + 338. 25s. net. 1928. (Blackie.)

Considered as a treatise on Invariant-Theory, this book has the merit of being an essentially new method of treating the subject; that is to say, it is in no sense a compilation or reproduction of previous works dealing with invariants. To build up his line of approach, the author starts with discussing the foundations of the elementary theory of determinants and of matrices; and from this basis he constructs his theory of invariants directly; in a measure, this process may be said to be novel for text-books in this country. At the same time it forms a mode of presentation which is peculiarly in keeping with the trend of certain tendencies in modern mathematical thought; and it leads the author directly to give some account of the work of writers such as Weitzenbock and Study.

In a subject with so many ramifications, great difficulty is found in knowing what to include and what to omit; but the author seems to show very sound judgment and skill in presenting the main lines of development, while at the same time he indicates where fuller details and more extended discussions may be found by those who wish to go further with the problems introduced here.

From the point of view of a rapid introduction to the theory of invariants, it is, perhaps, open to question whether the more elementary parts should not be studied before attacking the theory of matrices; but this book has the great advantage of reading as a connected whole, and it points forward to problems which call for investigation, suggesting various possible useful extensions of what has already been done.

Without adding greatly to the bulk, it might have been possible to include more details of the theorem on Canonical Forms discovered by the late E. K. Wakeford *; and an account of connected work, more recently developed by J. H. Grace, would also be welcomed by readers who have not the means of consulting easily the original sources.

Turning now to details of the theory of matrices, the book is almost exclusively based on the older English school of Cayley and Sylvester; hardly any use is made of the extensions found by Frobenius (in 1880), who worked with the equivalent idea of bilinear forms. For instance (ch. vi.), the theorem is proved that a matrix of n rows

$$A = \begin{Bmatrix} a_{11}, & a_{12}, & \dots, & a_{1n}, \\ a_{21}, & a_{22}, & \dots, & a_{2n}, \\ \cdot & \cdot & \cdot & \cdot, \\ a_{n1}, & a_{n2}, & \dots, & a_{nn}, \end{Bmatrix}$$

satisfies the equation $\Delta(A)=0$, where $\Delta(\lambda)$ is a polynomial of degree n in λ, found by the expansion of the determinant formed by subtracting λ from each element of the leading diagonal of the matrix. This result was given first by Hamilton (for three-rowed matrices); and was proved later by Cayley (perhaps independently) for the matrix of n rows. However, it is not true that $\Delta(A)=0$ is always the equation of *lowest* degree which the matrix satisfies; for example † the matrix

$$A = \begin{Bmatrix} 0, & 1, & 1 \\ 1, & 0, & 1 \\ 1, & 1, & 0 \end{Bmatrix}$$

leads to the determinant $\Delta(\lambda)=(\lambda+1)^2(2-\lambda)$, but the matrix satisfies not only the cubic $\Delta(A)=0$, but also the quadratic $(A+1)(A-2)=0$. In general Frobenius has proved that, when $\Delta(\lambda)$ has repeated factors, the

* Owing to Mr. Wakeford's sacrifice of himself in the War, his extraordinary powers in Geometry and Algebra were not so widely known as they deserve to be. I may therefore take an opportunity of referring briefly to the striking impression which he made on those who taught him; my own direct knowledge was in Applied Mathematics, which he seemed to regard as necessary but dull; still, if he took any interest in a particular problem, he could always provide a good solution. But those who dealt with his own subjects regarded his early loss as irreparable; and, in conversation with him, when he was in Cambridge shortly before his end, it was easy to see that he had a rare keenness in his work, much of which he succeeded in completing in the trenches.

† This matrix corresponds to the set of coefficients in a quadric surface of revolution; if the matrix is not restricted to be symmetrical, even simpler examples are easy to construct.

equation for the matrix A is found by dividing $\Delta(\lambda)$ by $F(\lambda)$, the H.C.F. of the first minors of $\Delta(\lambda)$.

In a cognate problem the reduced form of a linear substitution (given at the end of ch. xix.) needs to be reconsidered for the case of equal roots of the determinantal equation. For instance, the substitution

$$x' = 3x + y, \quad y' = -x + y$$

may be represented by the matrix

$$\begin{Bmatrix} 3, & 1 \\ -1, & 1 \end{Bmatrix}$$

and has the determinantal equation $(\lambda - 2)^2 = 0$; but this substitution *cannot* be reduced to the usual simple form *

$$X' = 2X, \quad Y' = 2Y.$$

Similarly in the theory of two quadratic forms (ch. xx.) the case of repeated roots leads to complications (but here references are given to other books dealing with the method of invariant-factors). At first sight it may be felt that the case of repeated roots is rather *special*; but in metrical geometry (for example) a theory of quadric surfaces could hardly be called *complete* if it ignored entirely the paraboloids and surfaces of revolution; and in the theory of two conics it is not safe to exclude the possibility of contact at a *single* point.† Thus the statement (see the footnote on p. 302) that the four coefficients in $\Delta(\lambda)$ form a *complete* system of invariants must be subject to reservations when there are equal roots in question. The additional information can be obtained from suitable covariants; but perhaps more quickly from the theory of invariant-factors, originally developed by Sylvester and completed by Weierstrass and Kronecker.

The book contains numerous examples which are well chosen so as to illustrate the general theory; and in the course of the argument occasion is taken to give much interesting historical information on the earlier development of the subject, so that the student will derive real benefit from the study of this treatise.

T. J. I'A. BROMWICH.

Reziprokentafel aller ganzen Zahlen von 1 bis 10000. BY M. VAN HAAFTEN. Pp. xxiv + 50. Fl. 2·40. 1926. (Noordhoff, Groningen.)

This is a seven-figure table, without differences, clearly printed in large type. Each page has four columns of fifty entries, and the argument of every entry is printed at length. That is to say, there is not the least attempt to economise space, and the extreme limit of simplicity is attained. It is hard to believe that anyone who can use a table at all resents a little intelligent compression, but one never knows.

A well-written introduction includes a very thorough bibliography, based ultimately on the *Encyklopädie*. To this might be added that in Goodwyn's anonymous publication "A Tabular Series of Decimal Quotients for all the proper vulgar fractions, of which, when in their lowest terms, neither the numerator nor the denominator is greater than 1000" (J. M. Richardson, London, 1823), not only are the reciprocals of the first thousand numbers implicitly contained, but the first five hundred entries form necessarily a pure table of reciprocals, albeit the arguments run backwards from 1000 to 501. Goodwyn's table differs from most others in showing the point from which a decimal, if infinite, begins to recur.

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* Take $X = x + y$, and replace x by X; then the substitution becomes

$$X' = 2X, \quad y' = -X + 2y.$$

Now if we take Y as any linear function $lX + my$, we see that $Y' = -mX + 2Y$; and here the term in X *cannot be removed*; but it can be standardised by putting $m = -1$, and then the reduced form is

$$X' = 2X, \quad Y' = X + 2Y, \quad \text{with} \quad X = x + y, \quad Y = l(x+y) - y.$$

† The assumption that the usual standard form still persists when $\Delta(\lambda) = 0$ has equal roots, means, in the theory of two conics, that only *double* contact can be handled.

Four-Figure Tables. By the late C. Godfrey and A. W. Siddons. Pp. iv + 40. 2s. 6d. 1927. (Camb. Univ. Press.)

Mathematical Tables (Four Figures). Compiled by C. V. Durell. Pp. 40. 9d. n.d. (Bell.)

Four-Place Mathematical Tables with Forced Decimals. Compiled by F. S. Carey and S. F. Grace. Pp. 40. 1s. 1927. (Longmans, Green.)

(1) The tables of elementary functions compiled by Mr. Siddons and the late Prof. Godfrey, which have been reprinted many times since they were published in 1913, have now been completely reset. There are some minor changes, in the order of the tables and in the type, and arguments are printed at the foot as well as at the head of the columns. But the substantial change is the addition of a thumb-index, provided without reduction in the size of numerals by the use of a broad page. These tables are relatively expensive, but they are well bound, and the adoption of this device for facilitating their use should render them even more popular than they have been in the past.

(2) The interesting feature of Mr. Durell's compilation is that the entries are separated by horizontal spaces into blocks of four, arguments in a block running from 0 to 3, from 4 to 7, from 8 to 11, and so on. To a computer, an arrangement which brings 20 to the top of one block and 30 to the middle of another would be insufferable, but perhaps the very reduction of the mechanical element will diminish the risk of mistakes by the careless schoolboy. The printing is very clear, and the limp cover, which is of unusually coarse cloth, should be durable. The book is good and cheap.

(3) The pages of the tables designed at Liverpool have an elaborate appearance. The decimal figures are from one fount when the entry is in excess of the true value, from another when it is in defect, and from a third in the few cases in which it is exact. The first two founts are used also for the difference columns; these are conveniently placed in the centre of the page, and a simple device makes them in effect columns for five instead of for ten of the main columns. I do not know to what extent the practical man prefers refinements in four figures to five figures used mechanically. Economy of space is not to be despised, for half the labour of working with such a volume as Chambers is in turning over the pages. In this direction the limit was doubtless attained by old Oliver Byrne, who reduced the entries of the seven-figure logarithms of the first ten million numbers to ten folio double-pages, not by small print, but by a variety of typographical ingenuities. Mr. Grace and the late Prof. Carey have been quite successful in their less ambitious plan of making the most of four figures. And as they say, if their devices are ignored, there remain the ordinary tables which schoolboys can use. At the same time, a healthy boy wants to understand gadgets if they are put into his hands, and the mathematical master has more important things to explain than forced decimals.

This book is distinguished from the others also by the inclusion of exponential and hyperbolic functions and of natural logarithms, a difference which to my mind is more important for class use than an improvement in the order of accuracy obtainable. Practice in numerical integration which does not frequently proceed to the bitter end is an absurdity: any teacher who questions this assertion may try the experiment of setting a few examples to a class which he thinks will find them child's play. The absurdity is perhaps tolerable if suitable tables are not available. It is true that the necessary calculations can all be performed without difficulty by means of ordinary logarithms. This is not an excuse to be urged by compilers who print separate sine and cosine tables, for they at least can hardly plead that they are taking intelligence and industry for granted. But if subsidiary calculations have to be performed and corrected, the teacher may reasonably object to numerical examples on the ground that three-quarters of the time spent on them is spent irrelevantly. Any serviceable set of tables which removes this objection should be welcomed widely.

The printing is clear, and the book is ridiculously cheap. I have only two complaints to make and one misgiving to express. It may be necessary to

make it *possible* for pages containing explanations to be detached, but to insert these pages where they *must* be detached if one of the tables is to be used comfortably is a mistake. And one does not want to have to cut the pages of a book of tables. I shall be surprised to learn that the binding is proving satisfactory in use, for I suspect it of being stiff enough to sustain damage that it is limp enough to incur. E. H. N.

Vorlesungen über Algebra. By GUSTAV BAUER. Revised by LUDWIG BIEBERBACH. Fourth Edition. Pp. i-x; 1-334, with index. 20 RM. 1928. (Teubner.)

During the last thirty years of the nineteenth century Professor Bauer was in the habit of delivering courses of lectures on Algebra at the University of Munich. These came to be very much appreciated, and in due course were published in book form. Already in 1900 the author was eighty years old. Then he lived to see the publication of the work in 1903, but three years later at the age of eighty-five he died. The present work is the fourth edition of what is evidently a standard German text-book for students upon algebra and the theory of equations. In the original edition, as far as one can gather from the present prefaces, Professor Bauer was assisted by Dr. Karl Doehlemann, one of his colleagues at Munich; and during the past few decades Doehlemann has twice re-edited the work. Now that both he and the original author have died, it has fallen to Professor Bieberbach of Berlin, whose name is well known as that of a distinguished analyst, to bring the book up to date.

Before indicating what is actually new in this latest edition, it is best to state the contents as a whole. The book covers much the same ground as that of Burnside and Panton, whose work on the theory of equations is very familiar to us in this country. The algebraic equation is the central theme. All the fundamental properties of the equation are clearly explained, developed, and proved. These include the theorem that every equation has a root. Then follows a very interesting section on determinants, matrices, and orthogonal transformations. This takes us to page 98, with about fifty pages devoted to each of these two first sections. Next there is a shorter third section on combinatory analysis; and this contains a masterly proof of the rather troublesome theorem that every polynomial function, symmetric in the roots of an equation, can also be expressed as a polynomial in the coefficients. The way is prepared by enlarging on the principle of "lexicographical arrangement." It is a simple enough matter, but the actual notation is new to the present reviewer, and is worth special mention. Thus the author would write

$$(4332) > (4323), \quad (ab \ldots c) > (pq \ldots r),$$

to indicate that, for his purpose, a term $x^4y^3z^3t^2$ *precedes* $x^4y^3z^2t^3$; and, more generally, $x^a y^b \ldots t^c$ precedes $x^p y^q \ldots t^r$.

In the fourth, and longer section, numerical solutions of equations are discussed: and this is followed by a section on the relative situations of the roots.

The fifth section is devoted to the algebraic solution of equations. It is formally the most important part of the book, and is subdivided into eight chapters. The first few of these chapters deal with the cubic, quartic and cyclic equations; and the final pages give the Galois theory of groups, and a proof that the solution of equations, whose degree exceeds four, cannot be stated with a finite number of radicals—that it is in fact insoluble in the elementary sense. There follows a brief discussion of continued fractions.

To many a good mathematician the Galois theory and the group theory in general are difficult. Probably this is largely because the reader has to go a long way before he attains solid results. He can easily be lost in the outskirts of the theory. It is therefore a pleasure to find in this volume a remarkably careful and lucid introduction to the theory. There is a crispness in the definition of *isomorphic* groups which is admirable. A set of k operations $S_1, S_2, \ldots, S_k$ form a group when each combination $S_\lambda S_\mu$ exists, and can be identified as a member S_p, say, of this group. Then if $T_1, T_2, \ldots, T_k$ is another group, also with k members, the product $T_\lambda T_\mu$ is necessarily one of these members, T_q. The author now points out that such statements, in

effect, define the integers p and q as specific functions of λ and μ. So $p=\phi(\lambda, \mu)$, $q=\psi(\lambda, \mu)$. If the equality, $\phi(\lambda, \mu)=\psi(\lambda, \mu)$, is identically true for all suffix pairs λ, μ, then the groups are isomorphic. It is this frank dealing in suffixes λ, μ, rather than the members S and T themselves, which is so enlightening.

The chief novelties in this fourth edition appear to be the more extended treatment of determinants and matrices, the references to original sources of theorems, and the occasional use of the theory of functions. The treatment of determinants, which is extremely good, is rendered specially attractive by its twofold line of approach. A function of n vectors is defined by three characteristic properties; for example, if $n=3$, and a, b, c are vectors, the function is written $D(a, b, c)$; and the three properties are indicated by the following typical statements:

$$\text{(i)}\ D(a, b, c)=D(a, b+c, c);$$
$$\text{(ii)}\ D(a, \lambda b, c)=\lambda D(a, b, c);$$
$$\text{(iii)}\ D(e_1, e_2, e_3)=1.$$

Here λ is a numerical multiplier, and e_1, e_2, e_3 are certain units. It is shown that this broad definition *completely* characterizes the determinant whose n columns are specified by the n vectors.

The book is extremely well printed, with few obvious misprints. The only faults that deserve mention are trivial: some of the spacing of elements in a determinant is ambiguous, and the notation for transposition of matrices tends to be indistinct or even to drop out.

Altogether it is an interesting and most suggestive book.

H. W. TURNBULL.

Elementary Conditions of Human Variability. By R. DODGE. Pp. xii+107. \$1.50. 1927. (New York: Columbia University Press.)

This is the final report, with full data, of experiments made during 1915-17 and previously discussed in *The Journal of Experimental Psychology*, 1923. The work appears to have been carefully planned and executed, but the favourable impression made at first sight rather fades away on perusal of the report. The variability studied is not that between different human subjects, but between responses of the same subject to a given stimulus at different times and so with different previous experience, *e.g.* two knee-jerks at half-second interval, change of pulse with exercise. It is difficult for an outsider to judge the adequacy of experimental work in relation to a given problem, but the observations are of very limited range and the treatment of the results is unsatisfactory. The conclusions drawn are, no doubt, justifiable, but a statistical treatment of the data might reveal much more and would at any rate put the main conclusions on a sound basis. The hope expressed that these results may serve as standards for the investigation of changes with age, etc., can hardly be fulfilled without a much more thorough discussion.

G. SMEAL.

Elementary Analytical Geometry. By T. H. WARD HILL. 264 pp. 10s. 1928. (Mills & Boon.)

A careful, almost elaborate, elementary treatment of the straight line, a pair of straight lines, the circle, and the conic sections. The book ends with three pages on the application of the differential calculus and twenty pages on the general conic. An especial point is made of parametric coordinates.

Practical Measurement as an Introduction to Science. By H. R. CHARTER. Pp. 80. 2s. 6d. 1928. (Longmans, Green & Co.)

The author believes that boys of fourteen not only profit by the training and discipline involved in a course of Practical Measurement, but take a real interest in it if they see its relation to Science as a whole. He has provided in this book a description of practical work in measurement and in finding "density." Instruction is given for carrying out and recording the experiments in such detail that the pupil could work through the course with the minimum of help from a master.

F. C. B.

Geometry for Preparatory Schools. By F. W. WESTAWAY. Pp. xi + 202. 3s. 6d. 1928. (Blackie.)

Any boy who imagines that geometry is devoid of human interest should read this book and he will soon think otherwise. It is the work of a teacher who knows how to arouse interest by means of a wealth of practical applications. The author's main purpose is to give an intelligent knowledge of the elementary *facts* of geometry without attempting to develop the subject on rigorously deductive lines from first principles. While making all possible use of the boys' intuitions and of their knowledge of space-relations in practical life, both two-dimensional and three-dimensional, he makes frequent little excursions into the domain of deductive geometry, excursions which become longer and longer until, towards the end of the course, he is able to show "how to write out proofs" and "how to solve" riders.

After some discussion on fundamental concepts, the elementary constructions are developed by using the idea of symmetry about an axis. The appeal to intuition often presents the facts in an order the reverse of that which is required later in the deductive development. For instance, the angle properties of parallels are taken before the conditions of parallelism, the property of the external angle of a triangle is deduced from the angle-sum, the equality of angles in the same segment of a circle comes before the fact that the angle at the centre is double any angle at the circumference that stands on the same arc. Some of the definitions demand data which are more than sufficient for the construction of the figure—for instance, a rectangle is defined as a right-angled parallelogram, and it is stated that the angles of a rectangle are right angles by definition. There are a few obvious errors which will doubtless be corrected in a future edition, such as the numerical example "3 : 12 = 12 : 36," explaining the meaning of *third proportional*. But though there may be some details which one would like to alter, there can be no doubt that this stimulating volume is full of good things.

There is no division into chapters, but the extensive use of clarendon type makes the various topics stand out conspicuously, and the many statements in dark type, labelled "L," are intended to be learnt by heart. W. J. D.

Junior Geometry. By A. E. TWEEDY. Pp. x + 189. 2s. 3d. 1928. (Dent.)

This book is intended as a two years' course for beginners, and deals with the elementary constructions, parallels, the simple properties of the circle, congruent triangles, special forms of the quadrilateral, and rectilinear areas, in this order, finishing up with Pythagoras' Theorem. A special feature is the postponement of congruency. The beginner is introduced to a right angle by constructing P the middle point of XY and finding a point Z equidistant from X and Y, and is then told that XPZ is 90°. His attention is not called to the constructed properties which the figure possesses, but is concentrated on the fact that this construction gives a right angle. So, also, in bisecting an angle his attention is not called to the constructed properties of the figure, but to a property which he can scarcely understand arises in consequence of the construction. It is at least debatable whether this can be a good way of commencing the study of geometry. The book is well printed, and there is an abundance of easy exercises. W. J. D.

The Elementary Differential Geometry of Plane Curves. By R. H. FOWLER. Cambridge Tracts, No. 20. 2nd edition. Pp. viii + 105. 6s. 1929. (Camb. Univ. Press.)

This is little changed from the first edition (*Gazette*, vol. x. No. 148, p. 151, Oct. 1920; the commendation is heartily endorsed by the present reviewer). The sign in the definition of torsion has been wisely and silently reversed, and a few examples are added where the length of the page allowed. The second preface notices some new work of Neville and Pollard which ought to have been embodied; but even without this the reprint is welcome. H. P. H.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **H. S. Hall,** a collection of back numbers of the *Gazette*;

From Mr. **F. P. White,** text-books by I. A. Barnett, F. Bowman (2 vols.), H. S. Jones, A. Lafay, and H. B. Phillips (2 vols.), together with:

H. Bagchi	Geometrical Analysis	1926
A. Eagle	Fourier's Theorem and Harmonic Analysis	1925
M. Kraïtchik	Théorie des Nombres	1922
L. Lewent	Conformal Representation Translated from German by R. Jones and D. H. Williams. The work was completed after the author's death by W. Blaschke.	1925
G. H. Ling, G. Wentworth, and D. E. Smith	Projective Geometry	1922
L. L. Smail	Theory of Infinite Processes	1923

The following have been bought:

P. E. B. Jourdain	Principle of Least Action Three essays, historical and philosophical, reprinted from *The Monist.*	1913
F. W. Newman	Mathematical Tracts; II: Anticyclics These are the hyperbolic functions, but the author, a brother of the famous Cardinal, had a passion for using names and notation of his own invention; the introduction of a number of Hebrew letters as functional symbols gives the pages a strange appearance. *Newman's first set of tracts is already in the Library.*	1889

NOTICE.—*Prof. Neville is visiting South Africa this summer, and from June 20th till the beginning of October correspondence relating to the Library should be addressed to* Mr. F. Beames, 7 Mansfield Road, Reading, *who will have access to the shelves.*

WANTED.

The set of eight portraits comprised in *Portraits of Mathematicians,* Portfolio No. 3; edited by Prof. D. E. Smith, and published by the Open Court Con[pany]. The subjects were Thales, Pythagoras, Euclid, Archimedes, Descart[es], Newton, Napier, and Pascal.—F. C. Ade, M.A., Jordans, Mottingham La[ne], S.E. 9.

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AT THE UNIVERSITY PRESS, GLASGOW.

BOOKS RECEIVED, JOURNALS, ETC.

May, 1929.

Bell & Sons. *Junior Arithmetic Tests.* Pp. 36. 6d. 1928.

Cajori, F. C. *A History of Mathematical Notations.* Vol. I. *Notations in Elementary Mathematics.* Pp. 451. 25s. net. 1928. (Open Court.)

Carslaw, H. S. *A Historical Note on Gibbs' Phenomenon in Fourier's Series and Integrals.* (Reprint. *Bull. Amer. Math. Soc.* vol. 31, No. 8, Oct. 1925.)

Carslaw, H. S. *Gibbs' Phenomenon in Fourier's Integrals.* (Reprint. *Nature* 116, Aug. 29, 1925.)

Carslaw, H. S. *Gibbs' Phenomenon in the Sum (C, r) for $r > 0$ of Fourier's Integrals.* (Reprint. *Journal L.M.S.* I. part i.)

(All in the Sydney University Reprints, Series XI.)

Charter, H. R. *Practical Measurement as an Introduction to Science.* Pp. 80. 2s. 6d. 1928. (Longmans, Green.)

Courant, R. *Vorlesungen über Differentiel- und Integralrechnung.* Vol. II. *Funktionen mehrerer Veränderlicher.* Pp. viii + 360. Bound. RM. 18.60. 1929. (Springer, Berlin.)

Durell, C. V. *A Key to Elementary Geometry.* Pp. 80. 3s. 6d. net. 1928. (Bell & Sons.)

Durell, C. V., and **Siddons, A. W.** *Graph Book. An Exercise and Text Book.* Pp. 80. 1s. 9d. Stiff Boards, 2s.; Teacher's Edition, with Hints and Answers, 2s. 6d. 1929. (Bell & Sons.)

Fawdry, R. C., and **Beaven, H. C.** *Elementary Algebra for Schools.* Part I. Pp 216 + xxxvi. 3s. 6d. with Answers; 3s. without. 1929. (Blackie.)

Foster, P. F., and **Baker, J. F.** *Differential Equations of Engineering Science.* Pp. vi + 184. 12s. 6d. net. 1929. (Milford, Ox. Univ. Press.)

Fowler, R. H. *Statistical Mechanics: The Theory of the Properties of Matter in Equilibrium.* Pp. 570. 35s. net. 1929. (Cam. Univ. Press.)

Fowler, R. H. *The Elementary Differential Geometry of Plane Curves.* (No. 20, Cambridge Tracts.) Second Edition. Pp. viii + 105. 6s. net. 1929. (Cambridge University Press.)

Fraenkel, A. *Einleitung in die Mengenlehre.* Pp. xiv + 424. RM. 22.60. Bd. RM. 24. 1928. (Springer, Berlin.)

Gibbs, R. W. M. *Stage A Geometry.* Pp. viii + 109. 2s. 1929. (Black.)

Grant, F. L., and **Hill, A. M.** *Commercial Arithmetic.* (5th Edition.) Pp. x + 430. With Ans. 5s. net. 1929. (Longmans.)

Haines, A. H. *Surveying for Agricultural Students, Land Agents and Farmers.* New Edition. Pp. 210. 12s. 6d. net. 1929. (Longmans, Green.)

Hölder, O. *Die Arithmetik in strenger Begründung.* (2nd Edition.) RM. 3.60. 1929. (Springer, Berlin.)

Jeffreys, H. *The Earth. Its Origin, History, and Physical Constitution.* 2nd Edition. Pp. xi + 346. 20s. net. 1929. (Cam. Univ. Press.)

Konečný, M. *Contribution à l'Etude des Propriétés Projectives du Contact.* Pp. 19. n.p. 1928. (Faculté des Sciences de l'Université Masaryk, Brno, Kounicova 63.)

L'Institut international de Physique Solvay. *Rapports et discussions du 5e Conseil de Physique. Oct.* 1927. *Electrons et Photons.* Pp. viii + 289. 60 fr. 1929. (Gauthier-Villars.)

Loria, G. *Storia delle Matematiche.* Vol. I. *Antichità: Medio Evo: Rinascimento.* Pp. 497. L. 23. 1929. (Soc. Tip. Edit. Naz. Torino.)

Maclean, J. *Graphs and Statistics. Elementary Applications of Mathematical Methods.* Pp. xiii + 200. 4 Rs. 1926. (R. Govind, Carnac Road, Bombay, 2.)

Maclean, J. *Mathematics and Life. A Plea for Cooperation in an Educational Experiment.* Pp. 18. Reprint from *The Times of India.* April 1928.

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. (Teubner, Leipzig.)

American Journal of Mathematics. (Johns Hopkins Press, Baltimore.)

American Mathematical Monthly. (Math. Assoc. of America, Menasha.)

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Annals of Mathematics. (Princeton University Press, N.J.)

Anuario. (Universidad Nacional de La Plata.)

Boletin Matematico. (Dr. Baidaff, Belgrano 909, Buenos Aires.)

Bolletín del Seminario Matemático Argentino. (Perú 222, Bueonos Aires.)

Bollettino della Unione Matematica Italiana. (Zanichelli, Bologna.)

Bulletin of the American Mathematical Society. (Bowes & Bowes, Cambridge.)

Bulletin of the Calcutta Mathematical Society. (Calcutta Univ. Press.)

Contribución al Estudio de las Ciencias Físicas y Matemáticas (Universidad Nacional de la Plata.)

Jahresbericht der Deutschen Mathematiker-Vereinigung. (Teubner, Leipzig.)

Journal de la Société Physico-Mathématique de Léningrade.

Journal of the Mathematical Association of Japan. (I. Mori, Tokyo Higher Normal School for Girls.)

L'Enseignement Mathématique. (Gauthier-Villars.)

Mathematics Teacher. (Yonkers, N.Y.)

Memoria. (Universidad Nacional de La Plata.)

Monatshefte für Mathematik und Physik. (Leipzig Acad., Verlagsgesellschaft, M.B.H.)

Nieuw Archief voor Wiskunde. Nieuwe Opgaven. (P. Noordhoff Uitgever, Groningen.)

Nieuwe Opgaven.

Periodico di Matematiche. (Zanichelli, Bologna.)

Proceedings of the Edinburgh Mathematical Society. (Bell & Sons.)

Proceedings of the Physico-Mathematical Society of Japan. (Faculty of Science, Tôkyô Imperial University.)

Publicaciones de la Facultad de Ciencias Físico-Matemáticas Universidad Nacional de la Plata.

Revista de Ciencias. (Apartado 1979, Lima, Peru.)

Revista Matemática Hispano-Americana. (Soc. Mat. Española, Madrid.)

Revue Semestrielle des Publications Mathématiques. (Gauthier Villars.)

School Science and Mathematics. (Mount Morris, Chicago, Ill.)

Sitzungsberichte der Berliner Mathematischen Gesellschaft. (W. Fr. Kaestner, Gottingen.)

The Eugenics Review. (Macmillan.)

The Half-Yearly Journal of the Mysore University.

The Japanese Journal of Mathematics. (National Research Council) Imperial Academy House, Ueno Park, Tokyo.)

The Journal of the Indian Mathematical Society. (S. Varadachari, Madras.)

The Journal of the London Mathematical Society. (Hodgson.)

Unterrichtsblätter für Mathematik und Naturwissenschaften. (Salle, Berlin.)

Wiskundige Opgaven met de Oplossingen. (Noordhoff, Groningen.)

LONDON BRANCH REPORT FOR 1928.

THE membership stands at 116 full, 95 associate members. The meetings held during the year, viz.:

Jan. 28th.	Annual Meeting. Members' Topics.
Feb. 25th.	The Psychology of Mathematical Ability.—A. G. HUGHES.
Mar. 17th.	Presidential Address: The regular and half-regular solids.—W. C. FLETCHER.
Oct. 6th.	Mathematics and the backward child.—DR. DODD.
Nov. 10th.	What we should teach in graphs.—C. T. DALTRY.
Dec. 8th.	How do we justify the place that mathematics has hitherto held in the school curriculum?—J. KATZ.

have proved their interest both by the full attendances and the vigorous discussions they have produced.

They have been announced in the *Gazette*, the *L.C.C. Gazette*, the *A.M.A.*, and the *London Teacher*.

The innovation "Members' Topics" for the Annual Meeting proved very popular and is being continued.

The Library contains for reference text-books of algebra and geometry, and may be consulted on application to F. C. Boon.

The committee regard the year as very successful, and record their gratitude to all who have helped to make it so: (1) the readers of papers, (2) Mercers' School for the use of a room for Committee Meetings, (3) Bedford College for hospitality, the value of which to the Branch it would be difficult to assess.

F. C. BOON (Hon. Sec.).

THE Annual Meeting of the Branch was held at Bedford College on Saturday, 23rd February, 1929.

The officers for the year were elected as follows: President, Prof. Levy. Chairman, J. Katz. Vice-Chairmen, Miss R. H. King; W. E. Paterson. Treasurer, W. M. Roberts, 22 Westmount Road, S.E. 9. Secretaries, Miss F. A. Yeldham, 99 Grove Park, S.E. 5; F. C. Boon, 49 Idmiston Road, S.E. 27. Committee, E. J. Atkinson; Miss E. M. Barrat; W. G. Bickley; Miss E. G. Crowe; C. T. Daltry; Miss M. J. Griffith; Miss C. Hornby; S. Inman; Miss M. J. Parker; Dr. W. F. Sheppard; H. V. Styler; Miss W. M. Walter; Dr. Jessie White; Miss D. Yonge; Miss L. A. Zelensky.

The Branch elected as Honorary Vice-Presidents Dr. Nunn, W. C. Fletcher, Esq., Prof. A. Lodge, and Dr. W. F. Sheppard. By instituting this office and electing these vice-presidents, the Branch desired to express its appreciation of the great services rendered by them to the Branch over a long period of years. The Branch also passed a very hearty vote of thanks to Miss Zelensky, who was resigning the post of Secretary which she had held since 1921.

After the business the Branch proceeded to discuss members' topics: (1) Style in writing out geometry, (2) arrangement of logarithmic work, (3) proof of $\sin(A+B)$ formula. The discussion was animated, Messrs. Inman, Kearney, Daltry, Dr. Sheppard, Miss King, Messrs. Katz, Boon, Bickley, Atkinson and Roberts taking part.

THERE was a meeting on Saturday, 16th March, at Bedford College. 64 members were present, and Mr. J. Katz was in the Chair.

The Meeting discussed the Standard Form method of multiplication and division of decimals. In the discussion it was multiplication that was chiefly referred to:

Miss M. Punnett (London Day Training College) opened. She was opposed to the imposition of uniformity of method until it had been proved that a best method existed and that it must always remain the best. What method was best could not be settled by discussion but only by wide statistical investigation. The different methods used were all mathematically sound,

preference must be assigned on the grounds of efficiency. The objection to Standard Form was that it introduced a needless complication. Her own preference was for the method in which the units figure of the multiplier was placed under the right-hand digit of the multiplicand. It was not perfect, as there might be confusion as to whether the position of the point in multiplier or multiplicand determined the point in the product. If contracted methods were dropped, she saw no reason why the counting method for fixing the point should not be reintroduced as the simplest of all methods.

Mr. A. S. Grant (President of the Association of Head Masters of Preparatory Schools) had opposed the Standard Form in the *Mathematical Gazette* twenty-one years ago; he had been converted and remained a loyal adherent of Standard Form ever since. It was the only method in which every digit set down (including the multiplier) had its correct local value. It was the best method for the first steps in decimal work; but, when thoroughly grasped, should not remain the only one. The danger of confusion disappeared if the transforming step were properly stated.

Mr. C. V. Durell (Winchester College) maintained that the purpose of mechanical arithmetic was not to make pupils think; that purpose could be better served in other ways. When once a process had been put on a rational basis, the pupil was not expected to repeat the argument in each sum. The application of Standard Form demanded thought to distinguish between the application to multiplication and division, and this was wrong in a mechanical process. The justification of a mechanical method was results. Dr. Ballard's investigations tended to show that in speed and accuracy Standard Form was inferior to the method of fixing the point by counting places. The investigation was not wide enough to be conclusive, but he believed that further investigation would lead to the same conclusion.

In contracted methods a rough approximation would be found—in multiplication by the counting method, in division by Standard Form—and this would fix the point in the answer.

Mr. R. A. M. Kearney explained a method of multiplication in which the leading digit of the multiplier was placed under the units' digit of the multiplicand, and of division in which, by placing the divisor over the dividend, division was seen as the inverse of multiplication.

A number of other members joined in the discussion. No show of hands was taken on a preference of method. But a show of hands on actual procedure revealed 75 per cent. as practising Standard Form, 25 per cent. practising other methods.

F. C. Boon (Hon. Sec.).

N.E. BRANCH.

The Annual Meeting was held at Armstrong College, Newcastle-on-Tyne, on Saturday, 16th March, 1929.

There are now 62 members and 11 associate members, and several others have notified their intention of joining this year. Three meetings have been held—the inaugural meeting in May when our President (Mr. J. Strachan, H.M.I.) gave an address on "Recent Progress in Mathematical Teaching"—and two meetings in the autumn term.

At the first of these we welcomed Prof. W. P. Milne of Leeds, whose paper on the "History of the Equation" was greatly appreciated. We were entertained by Miss Gurney at the Church High School for the second meeting, when Mr. J. L. Burchnall, of Durham University, gave a paper on "Commonsense Inequalities." The balance sheet showed a balance of £7 10s. 6d.

After the business meeting Professor W. N. Curtis, A.R.C.Sc., D.Sc., Professor of Physics at Armstrong College, gave an interesting paper on "Modern Scientific Research."

The Mathematical Association.

PUBLICATIONS.

"**The Mathematical Gazette**" is issued six times a year. The price per copy (to non-members) is usually 2s. 6d. net.

The following Reports have been issued by the Association:

(1) Reports on the Teaching of Elementary Mathematics, 1902-1908 (Geometry, Arithmetic and Algebra, Elementary Mechanics, Advanced School Mathematics, the Course required for Entrance Scholarships at the Universities, Mathematics in Preparatory Schools), 6d. net. [Out of print.]

(2) Revised Report on the Teaching of Elementary Algebra and Numerical Trigonometry (1911), 3d. net.

(3) Report on the Teaching of Mathematics in Preparatory Schools (1907), 3d. net.

(4) Report on the Correlation of Mathematical and Science Teaching, by a Joint Committee of the Mathematical Association and the Science Masters' Association (1909, reprinted 1917), 6d. net.

(5) A General Mathematical Syllabus for Non-Specialists in Public Schools (1913), 2d. net. [Out of print.]

(6) Catalogue of Current Mathematical Journals, etc., with the names of the Libraries in which they may be found (1913), 40 pp., 2s. 6d. net.

(7) Report on the Teaching of Elementary Mathematics in Girls' Schools (1916), 1s. net.

(8) Report on the Teaching of Mechanics (*The Mathematical Gazette*, No. 137, Dec. 1918), 1s. 6d. net.

(9) Report on the Teaching of Mathematics in Public and Secondary Schools (*The Mathematical Gazette*, No. 143, Dec. 1919), 2s. net.

(10) Reports on the Teaching of Arithmetic and Algebra, Elementary Mechanics and Advanced School Mathematics (1922), 1s. net. Reprinted from (1).

(11) Report on the Teaching of Mechanics in Girls' Schools (1923), 6d. net.

(12) Report on the Teaching of Geometry in Schools (1923, 2nd ed. 1925), 2s. net.

(13) Report on the Teaching of Mathematics in Preparatory Schools (1924), 1s. net.

(14) Report on the Teaching of Mathematics to Evening Technical Students (1926), 1s. net.

(15) A List of Books Suitable for School Libraries (1926), 1s. net.

Such of these Reports as are not out of print may be obtained from Messrs. G. Bell & Sons, Ltd., or from the Secretaries.

THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto."—BACON (Preface, *Maxims of Law*).

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"The Mathematical Gazette" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.

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